Unique existence and boundary stabilization of a wave equation associated with an integral equation*[†]

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Abstract

The aim in this paper is to deal with a linear wave equation as an initial-boundary value problem in which the unknown function and the unknown boundary value satisfy an appropriate integral equation. The first step is to prove the unique existence of the problem under the weaker initial condition than what in [11], the proof is based on Galerkin approximation associated with a Schauder's fixed point theorem and some compact standards. The second part is devoted to the study of global existence and the decay of both unknown function and unknown boundary value.

1 Introduction

We study the following problem

$$u_{tt} - u_{xx} + F(u, u_t) = f(x,t), \quad 0 < x < 1, 0 < t < T, \tag{1.1}$$

$$u(0,t) = 0, \tag{1.2}$$

$$-u_x(1,t) = P(t), \tag{1.3}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \tag{1.4}$$

where $F(u, u_t) = Ku + \lambda u_t$, with K, λ are given constants and u_0, u_1, f are given functions satisfying conditions specified later, and the unknown

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function $u(x,t)$ and the unknown boundary value $P(t)$ satisfy the following integral equation

$$P(t) = u(1,t) + u_t(1,t) + g(t) - \int_0^t k(t-s)u(1,s)ds, \tag{1.5}$$

where g, k are given functions.

In [13] Santos studied the asymptotic behavior of the solution of problem (1.1), (1.2), (1.4) associated with a boundary condition of memory type at $x = 1$ as follows

$$u(1,t) + \int_0^t g(t-s)\mu(s)u_x(1,s)ds = 0, \quad t > 0. \tag{1.6}$$

To make such a difficult condition more simplified, Santos transformed (1.6) into (1.3), (1.5).

In [8], the problem (1.1)-(1.5) is formed from the problem (1.1)-(1.4) wherein, the unknown function $u(x,t)$ and the unknown boundary value $P(t)$ satisfy the following Cauchy problem for an ordinary differential equation as follows

$$\begin{cases} P''(t) + \omega^2 P(t) = hu_{tt}(1,t), & 0 < t < T, \\ P(0) = P_0, \quad P'(0) = P_1, \end{cases} \tag{1.7}$$

where $h \geq 0, \omega > 0, P_0, P_1$ are given constants.

In [1], N.T. An and N.D. Trieu have studied a special case of problem (1.1)-(1.4), (1.7) with $u_0 = u_1 = P_0 = 0$ and $F(u, u_t) = Ku + \lambda u_t$, with $K \geq 0, \lambda \geq 0$ are given constants. In the later case the problem (1.1)-(1.4) and (1.7) is a mathematical model describing the shock of a rigid body and a linear viscoelastic bar resting on a rigid base [1].

From (1.7) we represent $P(t)$ in terms of $P_0, P_1, \omega, h, u_{tt}(1,t)$ and then by integrating by parts, we have

$$P(t) = hu(1,t) - g(t) - \int_0^t k(t-s)u(1,s)ds, \tag{1.8}$$

where

$$g(t) = -\big(P_0 - hu_0(1)\big)\cos\omega t - \frac{1}{\omega}\big(P_1 - hu_1(1)\big)\sin\omega t, \tag{1.9}$$

$$k(t) = h\omega\sin\omega t. \tag{1.10}$$

In [2] Bergounioux, Long and Dinh studied problem (1.1), (1.4) with the mixed boundary conditions (1.2), (1.3) standing for

$$u_x(0,t) = hu(0,t) + g(t) - \int_0^t k(t-s)u(0,s)ds, \tag{1.11}$$

$$u_x(1,t) + K_1 u(1,t) + \lambda_1 u_t(1,t) = 0, \tag{1.12}$$

where

$$g(t) = \big(P_0 - hu_0(0)\big)\cos\omega t + \frac{1}{\omega}\big(P_1 - hu_1(0)\big)\sin\omega t, \tag{1.13}$$

$$k(t) = h\omega\sin\omega t. \tag{1.14}$$

for $h \geq 0, \omega > 0, P_0, P_1, K, \lambda, K_1, \lambda_1$ are given constants.

In [6], Long, Dinh and Diem obtained the unique existence, regularity and asymptotic expansion of the problem (1.1)-(1.4) in the case of $Q(t) = K_1 u(1,t) + \lambda_1 u_t(1,t), u_x(0,t) = P(t)$ where $P(t)$ satisfies (1.7) with $u_{tt}(1,t)$ is replaced by $u_{tt}(0,t)$.

In [11], Long, Ut and Truc gave the unique existence, stability, regularity in time variable and asymptotic expansion for the solution of problem (1.1)-(1.5) when $F(u, u_t) = Ku + \lambda u_t$. In this case, the problem (1.1)-(1.5) is the mathematical model describing a shock problem involving a linear viscoelastic bar.

The present paper consists of two main parts. In part 1, for $(u_0, u_1) \in H^1 \times L^2, K, \lambda \in \mathbb{R}, f \in L^1(0,T;L^2), g \in L^2(0,T), k \in L^1(0,T)$, we prove a theorem of existence and uniqueness of a weak solution (u, P) of problem (1.1)-(1.5). In the proof, we use the Galerkin approximation associated with a Schauder's fixed point theorem and some compact standards. The second part is devoted to the study of global existence and the decay of the solution $\big(u(t), P(t)\big)$ with respect to t. We first study the decay of the component $u(t)$ of the solution $\big(u(t), P(t)\big)$ under more restrictive conditions, namely $0 < |K| < (2-\varepsilon_1-3\varepsilon_2)/7, 0 < \lambda < 2-2\varepsilon_2-4|K|, f \in L^1(\mathbb{R}_+;L^2)\cap L^2(\mathbb{R}_+;L^2), g \in L^2(\mathbb{R}_+), k, h \in L^1(\mathbb{R}_+)$, $\int_0^{+\infty} e^{\alpha t} g^2(t)dt < +\infty$, $\int_0^{+\infty} e^{\alpha t}\bigg(\int_0^t k(t-s)ds\bigg)^2 dt < +\infty$, $\int_0^{+\infty} e^{\alpha t}\|f(t)\|^2 dt < +\infty$, for some $\alpha, \varepsilon_1, \varepsilon_2 > 0$, where $h(t) = \int_0^t |k(t-s)|ds$. Later by modifying some stronger assumptions, we obtain the decays of both $u(t)$ and $P(t)$ in which we also base on a theorem of unique existence for problem (1.1)-(1.5) published in [11]. The results obtained here may be considered as the generalizations of those in An and Trieu [1] and in Long, Dinh, Santos, Ut and Truc $\big([2], [4], [7] - [11], [13]\big)$.

2 Existence and uniqueness

Put $\Omega = (0,1), Q = \Omega \times (0,+\infty), Q_T = \Omega \times (0,T), T > 0$. We omit the definitions of usual function space: $C^m(\overline{\Omega})$,$L^p = L^p(\Omega)$, $W^{m,p}(\Omega)$. We denote $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$, $1 \leq p \leq \infty$, $m = 0, 1, ...$

The norm in L^2 is denoted by $\| \cdot \|$. We also denote by $\langle \cdot, \cdot \rangle$ the scalar product in L^2 or pair of dual scalar product of a continuous linear functional with an element of a function space. We denote by $\| \cdot \|_X$ the norm of a Banach space X and by X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of the real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt \right)^{1/p} < \infty \quad \text{for} \quad 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{esssup}_{0<t<T} \|u(t)\|_X \qquad \text{for} \quad p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t)$, $u''(t) = u_{tt}(t)$, $u_x(t)$, $u_{xx}(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.
We put

$$V = \{v \in H^1 : v(0) = 0\}, \tag{2.1}$$

$$a(u,v) = \left\langle \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \right\rangle = \int_0^1 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} dx. \tag{2.2}$$

Here V is a closed subspace of H^1 and on V, $\|v\|_{H^1}$ and $\|v\|_V = \sqrt{a(v,v)}$ are two equivalent norms.
We then have the following lemma.

Lemma 1. *The imbedding* $V \hookrightarrow C([0,1])$ *is compact and*

$$\|v\|_{C([0,1])} \leq \|v\|_V, \quad \forall v \in V. \tag{2.3}$$

The proof is straightforward and we omit the details.

Lemma 2. *There exists the Hilbert orthonormal base* $\{\widetilde{w}_j\}$ *of* L^2 *consisting of the eigenfunctions* $\{\widetilde{w}_j\}$ *corresponding to the eigenvalue* λ_j *such that*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lim_{j \to +\infty} \lambda_j = +\infty, \tag{2.4}$$

$$a(\widetilde{w}_j, v) = \lambda_j \langle \widetilde{w}_j, v \rangle \text{ for all } v \in V, j = 1, 2, \cdots. \tag{2.5}$$

Furthermore, the sequence $\{\widetilde{w}_j / \sqrt{\lambda_j}\}$ *is also the Hilbert orthonormal base of* V *with respect to the scalar product* $a(\cdot,\cdot)$.

The proof of Lemma 2 can be found in $\big[$[12], Theorem 6.2.1, p.137$\big]$, with $H = L^2$, and $V, a(\cdot,\cdot)$ as defined by (2.1), (2.2).
We make the following assumptions:
(H_1) $\quad u_0 \in V, u_1 \in L^2$,
(H_2) $\quad K, \lambda \in \mathbb{R}$,
(H_3) $\quad f \in L^1(0,T;L^2)$,
(H_4) $\quad g \in L^2(0,T), k \in L^1(0,T)$.
Then we have the following theorem.

Theorem 1. *Let* (H_1)-(H_4) *hold. For* $T > 0$, *problem* (1.1)-(1.5) *has a unique weak solution* (u,P) *satisfying*

$$\begin{cases} u \in L^\infty(0,T;V), u' \in L^\infty(0,T;L^2), \\ u(1,\cdot) \in L^\infty(0,T) \cap H^1(0,T), P \in L^2(0,T). \end{cases} \tag{2.6}$$

Remark 1. The assumptions $(H_1), (H_3), (H_4)$ are weaker than what in [11]. However the solution of the corresponding problem in [11] has better properties.

Proof. The proof consists of four steps.
Step 1. *Galerkin approximation.*
Supposing that $\{\omega_j\}$ is a denumerable base of V and orthonormal in L^2 as in Lemma 2 $\big(w_j = \widetilde{w}_j/\sqrt{\lambda_j}\big)$. We find the approximate solution of problem (1.1)-(1.5) as follows

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t)\omega_j, \tag{2.7}$$

where the coefficients c_{mj} satisfy the following system of ordinary linear differential equations

$$\begin{aligned} \langle u''_m(t), \omega_j\rangle + \langle u_{mx}(t), \omega_{jx}\rangle + P_m(t)\omega_j(1) + &\langle F\big(u_m(t), u'_m(t)\big), \omega_j\rangle \\ &= \langle f(t), \omega_j\rangle, \quad 1 \le j \le m, \end{aligned} \tag{2.8}$$

$$P_m(t) = u_m(1,t) + u'_m(1,t) + g(t) - \int_0^t k(t-s)u_m(1,s)ds, \tag{2.9}$$

with

$$\begin{cases} u_m(0) = u_{0m} = \sum\limits_{j=1}^{m} \alpha_{mj}\omega_j \to u_0 & \text{strongly in} \quad V, \\ u'_m(0) = u_{1m} = \sum\limits_{j=1}^{m} \beta_{mj}\omega_j \to u_1 & \text{strongly in} \quad L^2. \end{cases} \tag{2.10}$$

The existence of the solution of the above system will be given in the following lemma.

Lemma 3. *Let* $(H_1)-(H_4)$ *hold. For fixed* $T>0$, *the system* (2.8)-(2.10) *has solution* $c_m=(c_{m1},c_{m2},\cdots,c_{mm})$ *on an interval* $[0,T_m]\subset[0,T]$.

Proof. The mentioned system of ordinary equations is equivalent to

$$\begin{cases} c''_{mi}(t)+\displaystyle\sum_{j=1}^{m}\langle\omega_{jx},\omega_{ix}\rangle c_{mj}(t)+P_m(t)\omega_i(1)+\big\langle F\big(u_m(t),u'_m(t)\big),\omega_i\big\rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =\langle f(t),\omega_i\rangle,\quad 1\le i\le m, \\ P_m(t)=g(t)+\displaystyle\sum_{j=1}^{m}\left[c_{mj}(t)+c'_{mj}(t)-\int_0^t k(t-s)c_{mj}(s)ds\right]\omega_j(1), \\ c_{mi}(0)=\alpha_{mi},c'_{mi}(0)=\beta_{mi},\quad 1\le i\le m. \end{cases} \tag{2.11}$$

By replacing $(2.11)_2$ into $(2.11)_1$, after some rearrangements, the system (2.11) becomes

$$\begin{cases} c''_{mi}(t)+\displaystyle\sum_{j=1}^{m}\big[\langle w_{jx},w_{ix}\rangle+w_j(1)w_i(1)\big]c_{mj}(t)+\sum_{j=1}^{m}w_j(1)w_i(1)c'_{mj}(t) \\ \qquad -\displaystyle\sum_{j=1}^{m}w_j(1)w_i(1)\int_0^t k(t-s)c_{mj}(s)ds+\langle F\big(u_m(t),u'_m(t)\big),w_i\rangle \\ \qquad\qquad\qquad\qquad\qquad =-g(t)w_i(1)+\langle f(t),w_i\rangle,1\le i\le m, \\ c_{mi}(0)=\alpha_{mi},c'_{mi}(0)=\beta_{mi},\quad 1\le i\le m. \end{cases} \tag{2.12}$$

Omit the index m, we rewrite the system (2.12) as follows

$$\begin{cases} c''_{i}(t)+\displaystyle\sum_{j=1}^{m}\big[\langle w_{jx},w_{ix}\rangle+w_j(1)w_i(1)\big]c_{j}(t)+\sum_{j=1}^{m}w_j(1)w_i(1)c'_{j}(t) \\ \qquad -\displaystyle\sum_{j=1}^{m}w_j(1)w_i(1)\int_0^t k(t-s)c_{j}(s)ds+\langle F\big(u(t),u'(t)\big),w_i\rangle \\ \qquad\qquad\qquad\qquad\qquad =-g(t)w_i(1)+\langle f(t),w_i\rangle,1\le i\le m, \\ c_{i}(0)=\alpha_{i},c'_{i}(0)=\beta_{i},\quad 1\le i\le m. \end{cases} \tag{2.13}$$

Now we have the following system equivalent to system (2.13)

$$\begin{cases} c''_i(t)=F_{1i}\big(t,c(t),c'(t)\big)+\displaystyle\int_0^t k(t-s)F_{2i}\big(c(s)\big)ds, \\ c_i(0)=\alpha_i,c'_i(0)=\beta_i,\quad 1\le i\le m, \end{cases} \tag{2.14}$$

where $F_{1i} : [0,T] \times \mathbb{R}^{2m} \to \mathbb{R}, F_{2i} : \mathbb{R}^m \to \mathbb{R}, 1 \leq i \leq m,$

$$\begin{cases} F_{1i}\big(t, c(t), c'(t)\big) = - \displaystyle\sum_{j=1}^{m} \Big[\langle \omega_{jx}, \omega_{ix} \rangle + \omega_j(1)\omega_i(1) \Big] c_j(t) \\ \qquad\qquad\qquad\qquad - \displaystyle\sum_{j=1}^{m} \omega_j(1)\omega_i(1)c_j'(t) - g(t)\omega_i(1) + \langle f(t), \omega_i \rangle \\ \qquad\qquad\qquad\qquad - \Big\langle F\Big(\displaystyle\sum_{j=1}^{m} \omega_j c_j(t), \sum_{j=1}^{m} \omega_j c_j'(t) \Big), \omega_i \Big\rangle, \\ F_{2i}\big(c(t)\big) = \displaystyle\sum_{j=1}^{m} \omega_j(1)\omega_i(1)c_j(t), c \in \mathbb{R}^m, 1 \leq i \leq m. \end{cases} \tag{2.15}$$

Put

$$\big(Gc\big)_i(t) = F_{1i}\big(t, c(t), c'(t)\big) + \int_0^t k(t-s)F_{2i}\big(c(s)\big)ds, 0 \leq t \leq T. \tag{2.16}$$

Then we have the following system

$$\begin{cases} c_i''(t) = \big(Gc\big)_i(t), 0 \leq t \leq T, \\ c_i(0) = \alpha_i, c_i'(0) = \beta_i, 1 \leq i \leq m, \end{cases} \tag{2.17}$$

or

$$c_i(t) = \alpha_i + \beta_i(t) + \int_0^t d\tau \int_0^\tau \big(Gc\big)_i(s)ds \equiv \big(Uc\big)_i(t), 0 \leq t \leq T, 1 \leq i \leq m. \tag{2.18}$$

For $c = (c_1, \cdots, c_m) \in C^1\big([0,T]; \mathbb{R}^m\big)$, put $Uc = \big((Uc)_1, \cdots, (Uc)_m\big)$ where $(Uc)_i, 1 \leq i \leq m$, is as in (2.18). We prove that, for T is given, there exists $M > 0, T_m \in (0,T]$ such that $U : S \to S$ has at least one fixed point in $S = \{c \in C^1([0,T_m]; \mathbb{R}^m) : \|c\|_1 \leq M\}$, here the norm in $C^1([0,T_m]; \mathbb{R}^m)$ is defined as follows

$$\|c\|_1 = \|c\|_0 + \|c'\|_0, \|c\|_0 = \sup_{0 \leq t \leq T_m} |c(t)|_1, |c(t)|_1 = \sum_{i=1}^{m} |c_i(t)|. \tag{2.19}$$

- *Mapping* $U : C^1([0,T_m]; \mathbb{R}^m) \to C^1([0,T_m]; \mathbb{R}^m)$ *is continuous.*
  For $c \in C^1([0,T_m]; \mathbb{R}^m)$, we have

$$\big(Uc\big)_i'(t) = \beta_i + \int_0^t F_{1i}\big(s, c(s), c'(s)\big)ds + \int_0^t d\tau \int_0^\tau k(\tau - s)F_{2i}\big(c(s)\big)ds. \tag{2.20}$$

From (2.18)-(2.20) and assumptions $(H_3), (H_4)$, we deduce that $(Uc)'_i$ is continuous on $[0, T_m]$. Hence, $U : C^1([0, T_m]; \mathbb{R}^m) \to C^1([0, T_m]; \mathbb{R}^m)$ is well defined.

For $\{c_\nu\} \subset C^1([0, T_m]; \mathbb{R}^m)$ such that $c_\nu \to c$ in $C^1([0, T_m]; \mathbb{R}^m)$, we have

$$\begin{cases} (Uc_\nu)_i(t) - (Uc)_i(t) = \int_0^t d\tau \int_0^\tau \big[(Gc_\nu)_i(s) - (Gc)_i(s)\big] ds, \\ (Uc_\nu)'_i(t) - (Uc)'_i(t) = \int_0^t \big[(Gc_\nu)_i(s) - (Gc)_i(s)\big] ds. \end{cases} \tag{2.21}$$

Then we deduce that

$$\|Uc_\nu - Uc\|_1 \le \big(T_m^2 + T_m\big) \|Gc_\nu - Gc\|_0. \tag{2.22}$$

Besides, we have

$$\begin{aligned} (Gc_\nu)_i(t) - (Gc)_i(t) =& F_{1i}\big(t, c_\nu(t), c'_\nu(t)\big) - F_{1i}\big(t, c(t), c'(t)\big) \\ &+ \int_0^t k(t-s) F_{2i}\big[c_\nu(s) - c(s)\big] ds. \end{aligned} \tag{2.23}$$

In addition, we also have

$$\begin{cases} \sup\limits_{0 \le t \le T_m} \sum\limits_{i=1}^m \big|F_{1i}\big(t, c_\nu(t), c'_\nu(t)\big) - F_{1i}\big(t, c(t), c'(t)\big)\big| \\ \qquad \le \delta_1 \|c_\nu - c\|_0 + \delta_2 \|c'_\nu - c'\|_0 + |K| \|c_\nu - c\|_0 + |\lambda| \|c'_\nu - c'\|_0, \\ \sup\limits_{0 \le t \le T_m} \sum\limits_{i=1}^m \left| \int_0^t k(t-s) F_{2i}\big[c_\nu(s) - c(s)\big] ds \right| \le \delta_3 \|k\|_{L^1(0,T)} \|c_\nu - c\|_0, \end{cases} \tag{2.24}$$

where positive constants $\delta_i (i = 1, 2, 3)$ only depend on $\omega_i (i = \overline{1, m})$ and c.

From (2.22)-(2.24), we deduce that U is continuous on $C^1([0, T_m]; \mathbb{R}^m)$.

- *Now we show that U maps S into itself.*

For $c \in S$, we have

$$\begin{cases} \big(Uc\big)_i(t) = \alpha_i + \beta_i t + \int_0^t d\tau \int_0^\tau \big(Gc\big)_i(s) ds, 0 \le t \le T_m, 1 \le i \le m, \\ \big(Uc\big)'_i(t) = \beta_i + \int_0^t \big(Gc\big)_i(s) ds, 0 \le t \le T_m, 1 \le i \le m. \end{cases} \tag{2.25}$$

Then we deduce from (2.25), that

$$\begin{cases} \left|\left(Uc\right)(t)\right|_1 \leq \sum\limits_{i=1}^{m}|\alpha_i| + T_m \sum\limits_{i=1}^{m}|\beta_i| + T_m\left[\gamma_T + T_m\beta(M,T)\right], \\ \left|\left(Uc\right)'(t)\right|_1 \leq \sum\limits_{i=1}^{m}|\beta_i| + \gamma_T + T_m\beta(M,T), \end{cases} \tag{2.26}$$

where

$$\begin{cases} \beta(M,T) = \sum\limits_{i=1}^{m}\left[N_1(F_{1i},M) + \|k\|_{L^1(0,T)}N_2(F_{2i},M)\right], \\ \gamma_T = \sum\limits_{i=1}^{m}\int_0^T \left[|g(t)||\omega_i(1)| + |\langle f(t),\omega_i\rangle|\right]dt, \\ N_1(F_{1i},M) = \sup\left\{\sum\limits_{j=1}^{m}\left|\left[\langle\omega_{jx},\omega_{ix}\rangle + \omega_j(1)\omega_i(1)\right]c_j(t)\right| \right. \\ \qquad\qquad + \sum\limits_{j=1}^{m}\left|\omega_j(1)\omega_i(1)c_j'(t)\right| + \left|\left\langle F\left(\sum\limits_{j=1}^{m}\omega_j c_j(t),\right.\right.\right. \\ \qquad\qquad \left.\left.\left.\sum\limits_{j=1}^{m}\omega_j c_j'(t)\right),\omega_i\right\rangle\right| : \|c\|_{\mathbb{R}^m} \leq M, \|c'\|_{\mathbb{R}^m} \leq M\right\}, \\ N_2(F_{2i},M) = \sup\left\{\left|\sum\limits_{j=1}^{m}\omega_j(1)\omega_i(1)c_j(t)\right| : \|c\|_{\mathbb{R}^m} \leq M\right\}. \end{cases} \tag{2.27}$$

Hence, we have from (2.26), (2.27), that

$$\|Uc\|_1 \leq \sum_{i=1}^{m}\left(|\alpha_i|+|\beta_i|\right)+\gamma_T+T_m\left[\sum_{i=1}^{m}|\beta_i|+\gamma_T+(1+T)\beta(M,T)\right]. \tag{2.28}$$

By choosing $M \geq 2\left[\sum\limits_{i=1}^{m}\left(|\alpha_i| + |\beta_i|\right) + \gamma_T\right]$, and $T_m \in (0,T]$ such that $T_m\left[\sum\limits_{i=1}^{m}|\beta_i| + \gamma_T + (1+T)\beta(M,T)\right] \leq \frac{M}{2}$. Hence, $\|Uc\|_1 \leq M$ for all $c \in S$, that is, the operator U maps S the set into itself.

- *Next, we show that the set* $\overline{US}$ *is a compact subset of* $C^1([0,T_m];\mathbb{R}^m)$.

Since $US \subset S$, $US = \{Uc, c \in S\}$ is bounded in $C^1([0,T_m];\mathbb{R}^m)$. Now we have to prove US are equicontinuous in $C^1([0,T_m];\mathbb{R}^m)$.

For $c \in S, t, t' \in [0, T_m]$, we have

$$\begin{cases} \left(Uc\right)_i(t) - \left(Uc\right)_i(t') = \beta_i(t-t') + \displaystyle\int_{t'}^{t} d\tau \int_0^{\tau} \left(Gc\right)_i(s)ds, \\ \left(Uc\right)_i'(t) - \left(Uc\right)_i'(t') = \displaystyle\int_{t'}^{t} \left(Gc\right)_i(s)ds. \end{cases} \tag{2.29}$$

Hence, it follows that

$$\begin{cases} \left|\left(Uc\right)(t) - \left(Uc\right)(t')\right|_1 \leq \left[\displaystyle\sum_{i=1}^{m} |\beta_i| + \gamma_T + T_m\beta(M,T)\right]|t-t'|, \\ \left|\left(Uc\right)'(t) - \left(Uc\right)'(t')\right|_1 \leq \Big[\eta_T + \beta(M,T)\Big]|t-t'|, \end{cases} \tag{2.30}$$

where

$$\eta_T = \sup_{0 \leq t \leq T_m} \sum_{i=1}^{m} \big[|g(t)||\omega_i(1)| + |\langle f(t), \omega_i\rangle|\big]. \tag{2.31}$$

From (2.30), we deduce that US is equicontinuous in $C^1([0,T_m];\mathbb{R}^m)$ with respect to the norm $\|\cdot\|_1$. Applying Arzela-Ascoli's theorem, we conclude that $\overline{US}$ is compact in $C^1([0,T_m];\mathbb{R}^m)$. By the Schauder fixed-point theorem, we conclude that U has a fixed point $c \in S$. Hence, the proof of Lemma 3 is complete. □

Using Lemma 3, for $T > 0$, fixed, system (2.8)-(2.10) has solution $\big(u_m(t), P_m(t)\big)$ on an interval $[0,T_m] \subset [0,T]$. The following estimates allow one to take $T_m = T$ for all m.

Step 2. *A priori estimates.* Substituting (2.9) into (2.8), then multiplying the j^{th} equation of (2.8) by $c'_{mj}(t)$ and summing up with respect to j, afterwards, integrating by parts with respect to the time variable from 0 to t, we get after some rearrangements

$$\begin{aligned} S_m(t) = &S_m(0) - 2\int_0^t g(s)u_m'(1,s)ds \\ &+ 2\int_0^t u_m'(1,s)\left(\int_0^s k(s-\tau)u_m(1,\tau)d\tau\right)ds \\ &- 2K\int_0^t \langle u_m(s), u_m'(s)\rangle ds - 2\lambda\int_0^t \|u_m'(s)\|^2 ds \\ &+ 2\int_0^t \langle f(s), u_m'(s)\rangle ds \end{aligned} \tag{2.32}$$

where

$$S_m(t) = \|u'_m(t)\|^2 + \|u_{mx}(t)\|^2 + u_m^2(1,t) + 2\int_0^t |u'_m(1,s)|^2 ds. \tag{2.33}$$

Using the inequality

$$2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2, \forall a, b \in \mathbb{R}, \text{ for all } \varepsilon > 0, \tag{2.34}$$

and the following inequality

$$|u_m(1,t)| \leq \|u_m(t)\|_{C^0(\overline{\Omega})} \leq \|u_{mx}(t)\| \leq \sqrt{S_m(t)}, \tag{2.35}$$

we shall estimate the following terms on the right-hand side of (2.32).
First integral.

$$\begin{aligned} I_1 = -2\int_0^t g(s) u'_m(1,s) ds &\leq \frac{1}{\varepsilon}\int_0^t g^2(s) ds + \varepsilon \int_0^t |u'_m(1,s)|^2 ds \\ &\leq \frac{1}{\varepsilon}\int_0^t g^2(s) ds + \frac{\varepsilon}{2} S_m(t). \end{aligned} \tag{2.36}$$

Second integral.

$$\begin{aligned} I_2 &= 2\int_0^t u'_m(1,s)\left(\int_0^s k(s-\tau) u_m(1,\tau) d\tau\right) ds \\ &\leq \varepsilon \int_0^t |u'_m(1,s)|^2 ds + \frac{1}{\varepsilon}\int_0^t \left(\int_0^s k(s-\tau) u_m(1,\tau) d\tau\right)^2 ds \\ &\leq \frac{\varepsilon}{2} S_m(t) + \frac{1}{\varepsilon}\int_0^t \left(\int_0^s |k(s-\tau)| d\tau\right)\left(\int_0^s |k(s-\tau)| u_m^2(1,\tau) d\tau\right) ds \\ &\leq \frac{\varepsilon}{2} S_m(t) + \frac{1}{\varepsilon}\|k\|_{L^1(0,T)} \int_0^t \left(\int_0^s |k(s-\tau)| u_m^2(1,\tau) d\tau\right) ds. \end{aligned} \tag{2.37}$$

By inverting the variables of integration between s and τ for the integral in the right-hand side of (2.37), we get

$$\begin{aligned} I_2 &\leq \frac{\varepsilon}{2} S_m(t) + \frac{1}{\varepsilon}\|k\|_{L^1(0,T)} \int_0^t u_m^2(1,\tau) d\tau \left(\int_\tau^t |k(s-\tau)| ds\right) ds \\ &\leq \frac{\varepsilon}{2} S_m(t) + \frac{1}{\varepsilon}\|k\|^2_{L^1(0,T)} \int_0^t S_m(s) ds. \end{aligned} \tag{2.38}$$

Third integral.

$$
\begin{aligned}
I_3 = -2K\int_0^t \langle u_m(s), u_m'(s)\rangle ds \leq 2|K|\int_0^t \langle |u_m(s)|, |u_m'(s)|\rangle ds& \\
\leq 2|K|\int_0^t \|u_{mx}(s)\|\|u_m'(s)\|ds& \qquad (2.39)\\
\leq 2|K|\int_0^t S_m(s)ds.&
\end{aligned}
$$

Fourth integral.

$$
I_4 = -2\lambda\int_0^t \|u_m'(s)\|^2 ds \leq 2|\lambda|\int_0^t S_m(s)ds. \qquad (2.40)
$$

Fifth integral.

$$
\begin{aligned}
I_5 = 2\int_0^t \langle f(s), u_m'(s)\rangle ds \leq \int_0^t \|f(s)\|\|u_m'(s)\|ds& \\
\leq \int_0^t \|f(s)\|ds + \int_0^t \|f(s)\|\|u_m'(s)\|^2 ds& \qquad (2.41)\\
\leq \int_0^t \|f(s)\|ds + \int_0^t \|f(s)\|S_m(s)ds.&
\end{aligned}
$$

Since (H_1), we deduce the existence of positive constant M such that

$$
S_m(0) \leq \|u_{1m}\|^2 + \|u_{0mx}\|^2 + u_{0m}^2(1) \leq M. \qquad (2.42)
$$

Combining (2.32),(2.36)-(2.42), we obtain

$$
\begin{aligned}
S_m(t) \leq& \varepsilon S_m(t) + \frac{1}{\varepsilon}\int_0^t g^2(s)ds + \int_0^t \|f(s)\|ds + M \\
&+ \int_0^t \left[\frac{1}{\varepsilon}\|k\|_{L^1(0,T)}^2 + 2|K| + 2|\lambda| + \|f(s)\|\right] S_m(s)ds.
\end{aligned} \qquad (2.43)
$$

By choosing $\varepsilon = \frac{1}{2}$, we deduce that

$$
\begin{aligned}
S_m(t) \leq& 2M + 2\int_0^t \|f(s)\|ds + 4\int_0^t g^2(s)ds \\
&+ \int_0^t \left[4\|k\|_{L^1(0,T)}^2 + 4|K| + 4|\lambda| + \|f(s)\|\right] S_m(s)ds.
\end{aligned} \qquad (2.44)
$$

From assumptions $(H_3), (H_4)$, there exists positive constant C_1 such that

$$2M + 2\int_0^t \|f(s)\|ds + 4\int_0^t g^2(s)ds \le C_1, \forall t \in [0,T]. \tag{2.45}$$

Then we receive from (2.44) and (2.45), that

$$S_m(t) \le C_1 + \int_0^t \left[4\|k\|^2_{L^1(0,T)} + 4|K| + 4|\lambda| + \|f(s)\|\right] S_m(s)ds. \tag{2.46}$$

By Gronwall's inequality and assumption (H_3), we obtain

$$\begin{aligned} S_m(t) \le & C_1 \exp\left(4t\|k\|^2_{L^1(0,T)} + 4t|K| + 4t|\lambda| + \int_0^t \|f(s)\|ds\right) \\ & \equiv C_T \text{ for all } t \in [0,T]. \end{aligned} \tag{2.47}$$

In addition, we have

$$P_m^2(t) \le 8\left[u_m^2(1,t) + |u_m'(1,t)|^2 + g^2(t) + \left(\int_0^t k(t-s)u_m(1,s)ds\right)^2\right]. \tag{2.48}$$

Hence we get

$$\begin{aligned} \int_0^T P_m^2(t)dt \le & 8\left[\int_0^T u_m^2(1,t)dt + \int_0^T |u_m'(1,t)|^2 dt + \int_0^T g^2(t)dt \right. \\ & \left. + \int_0^T \left(\int_0^t k(t-s)u_m(1,s)ds\right)^2 dt\right]. \end{aligned} \tag{2.49}$$

Then, from (2.49) and assumption (H_4), we deduce

$$\|P_m\|^2_{L^2(0,T)} \le M_T, \tag{2.50}$$

where

$$M_T = 8\left(TC_T + \frac{C_T}{2} + TC_T\|k\|^2_{L^1(0,T)} + \int_0^T g^2(t)dt\right). \tag{2.51}$$

Step 3. *Limiting process.* From (2.33), (2.47), (2.50), we deduce the existence of a subsequence of $\{(u_m, P_m)\}$ still also so denoted, such that

$$\begin{cases} u_m \to u & \text{in} & L^\infty(0,T;V) & \text{weak}^\star, \\ u_m' \to u' & \text{in} & L^\infty(0,T;L^2) & \text{weak}^\star, \\ u_m(1,\cdot) \to u(1,\cdot) & \text{in} & L^\infty(0,T) & \text{weak}^\star, \\ u_m(1,\cdot) \to u(1,\cdot) & \text{in} & H^1(0,T) & \text{weakly}, \\ P_m \to P & \text{in} & L^2(0,T) & \text{weakly}. \end{cases} \tag{2.52}$$

By compact lemma of Lions $\big[[5], \text{p.}57\big]$ and the imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$, we're able to deduce from (2.52) the existence of a subsequence still denoted by $\{(u_m, P_m)\}$ such that

$$\begin{cases} u_m \to u \quad \text{in} \quad L^2(Q_T) \quad \text{strongly}, \\ u_m(1,\cdot) \to u(1,\cdot) \quad \text{in} \quad C^0\big([0,T]\big) \quad \text{strongly}. \end{cases} \tag{2.53}$$

From (2.9), $(2.52)_4$, we have

$$P_m(t) \to u(1,t) + u'(1,t) - \int_0^t k(t-s)u(1,s)ds + g(t) \equiv \widetilde{P}(t) \tag{2.54}$$

in $L^2(0,T)$ weakly.
Then we deduce from $(2.52)_5$, (2.54), that

$$P(t) \equiv \widetilde{P}(t). \tag{2.55}$$

Passing to the limit in (2.8)-(2.10) by (2.52)-(2.55), we have (u, P) satisfying the problem

$$\frac{d}{dt}\langle u'(t), v\rangle + \langle u_x(t), v_x\rangle + P(t)v(1) + \langle Ku(t) + \lambda u'(t), v\rangle \\ = \langle f(t), v\rangle, \forall v \in V, \tag{2.56}$$

$$u(0) = u_0, u'(0) = u_1, \tag{2.57}$$

$$P(t) = u(1,t) + u'(1,t) - \int_0^t k(t-s)u(1,s)ds + g(t). \tag{2.58}$$

Step 4. *The uniqueness of solution.* Let u_1, u_2 be two weak solution of problem (1.1)-(1.5) such that

$$\begin{cases} u_i \in L^\infty(0,T;V), u_i' \in L^\infty(0,T;L^2), \\ u(1,\cdot) \in L^\infty(0,T) \cap H^1(0,T), P_i \in L^2(0,T). \end{cases} \tag{2.59}$$

Then (u, P) with $u = u_1 - u_2$ and $P = P_1 - P_2$ satisfy the variational problem

$$\begin{cases} \langle u''(t), v\rangle + \langle u_x(t), v_x\rangle + P(t)v(1) + \langle Ku + \lambda u', v\rangle = 0, \forall v \in V, \\ u(0) = u'(0) = 0, \end{cases} \tag{2.60}$$

where

$$P(t) = u(1,t) + u'(1,t) - \int_0^t k(t-s)u(1,s)ds. \tag{2.61}$$

By taking $v = u'$ in $(2.60)_1$, then integrating in t, we get

$$\begin{aligned} S(t) =& 2\int_0^t u'(1,s)\Big(\int_0^s k(s-\tau)u(1,\tau)d\tau\Big)ds \\ &- 2K\int_0^t \langle u(s), u'(s)\rangle ds - 2\lambda\int_0^t \|u'(s)\|^2 ds, \end{aligned} \tag{2.62}$$

where

$$S(t) = \|u'(t)\|^2 + \|u_x(t)\|^2 + u^2(1,t) + 2\int_0^t |u'(1,s)|^2 ds. \tag{2.63}$$

Now we have some following estimations

$$S(t) \geq \|u'(t)\|^2 + \|u_x(t)\|^2 + 2\int_0^t |u'(1,s)|^2 ds, \tag{2.64}$$

$$|u(1,t)| \leq \|u(t)\|_{C^0(\overline{\Omega})} \leq \|u_x(t)\| \leq \sqrt{S(t)}. \tag{2.65}$$

By using (H_4) and inequalities (2.64), (2.65), we deduce from (2.62), that

$$S(t) \leq \frac{1}{2}S(t) + \big(2|K| + 2|\lambda| + \|k\|^2_{L^1(0,T)}\big)\int_0^t S(s)ds. \tag{2.66}$$

Hence we obtain

$$S(t) \leq N_T\int_0^t S(s)ds, \tag{2.67}$$

where

$$N_T = 2|K| + 2|\lambda| + \|k\|^2_{L^1(0,T)}. \tag{2.68}$$

By Gronwall's inequality, we deduce that

$$S(t) \equiv 0. \tag{2.69}$$

In summary, Theorem 1 has been completely proved. □

3 Decay of solution

In this part we will consider the solution of problem (1.1)-(1.5) in the sense of global existence and asymptotic behavior when $t \to +\infty$. We first modify some following assumptions.

$(\widetilde{H}_2)$ $\lambda \in \mathbb{R}_+, K \in \mathbb{R} : 0 < |K| \leq \frac{1}{2} - \varepsilon, \varepsilon > 0,$

$(\widehat{H}_2)$ $\lambda \in \mathbb{R}_+, K \in \mathbb{R}$ such that

$$\begin{cases} 0 < |K| < \frac{2-\varepsilon_1 - 3\varepsilon_2}{7}, \\ 0 < \lambda < 2 - 2\varepsilon_2 - 4|K|, \end{cases}$$

for some $\varepsilon_1, \varepsilon_2 > 0$, and note that $(\widehat{H}_2)$ is a special case of $(\widetilde{H}_2)$ for some $\varepsilon, \varepsilon_1, \varepsilon_2 > 0$ suitable,

$(\widetilde{H}_3)$ $f \in L^1(\mathbb{R}_+; L^2) \cap L^2(\mathbb{R}_+; L^2),$

$(\widetilde{H}_4)$ $g \in L^2(\mathbb{R}_+); k, h \in L^1(\mathbb{R}_+)$, where $h(t) = \int_0^t |k(t-s)|ds,$

$(\widetilde{H}_5)$ There exists a constant $\alpha > 0$ such that

$$\begin{cases} \int_0^{+\infty} e^{\alpha t} g^2(t)dt < +\infty, \\ \int_0^{+\infty} e^{\alpha t} \Big(\int_0^t k(t-s)ds \Big)^2 dt < +\infty, \\ \int_0^{+\infty} e^{\alpha t} \|f(t)\|^2 dt < +\infty. \end{cases}$$

Under assumptions $(H_1), (\widetilde{H}_2) - (\widetilde{H}_4)$ and for any $T > 0$, by Theorem 1, problem (1.1)-(1.5) has a unique weak solution (u, P) satisfying

$$\begin{cases} u \in L^\infty(0,T;V), u' \in L^\infty(0,T;L^2), \\ u(1,\cdot) \in L^\infty(0,T) \cap H^1(0,T), P \in L^2(0,T). \end{cases} \tag{3.1}$$

Then we have the following lemma.

Lemma 4. *Suppose that* $(H_1), (\widetilde{H}_2) - (\widetilde{H}_4)$ *hold. Then there is a unique solution* $\big(u(t), P(t)\big)$ *of problem* (1.1)-(1.5) *defined on* $\mathbb{R}_+$.
Moreover,

$$\|u'(t)\|^2 + \|u_x(t)\|^2 \leq C, \forall t \geq 0, \tag{3.2}$$

where C *is a positive constant depending only on* u_0, u_1, g, k, f.

Proof. By multiplying the equation (1.1) by u_t and integrating over $(0,1) \times (0,t)$, we obtain

$$\begin{aligned} \widetilde{S}(t) =& \widetilde{S}(0) + K\|u(t)\|^2 - K\|u(0)\|^2 - 2\int_0^t g(s)u'(1,s)ds \\ &+ 2\int_0^t u'(1,s)\Big(\int_0^s k(s-\tau)u(1,\tau)d\tau\Big)ds \\ &+ 2\int_0^t \langle f(s), u'(s)\rangle ds \end{aligned} \tag{3.3}$$

where

$$\begin{aligned}\widetilde{S}(t) =&\|u'(t)\|^2 + \|u_x(t)\|^2 + u^2(1,t) + 2\int_0^t |u'(1,s)|^2 ds \\ &+ 2\lambda \int_0^t \|u'(s)\|^2 ds.\end{aligned} \tag{3.4}$$

Using the same method of estimating in the proof of Theorem 1, we have that

$$J_1 = -2\int_0^t g(s)u'(1,s)ds \le \frac{1}{\varepsilon}\int_0^t g^2(s)ds + \frac{\varepsilon}{2}\widetilde{S}(t), \tag{3.5}$$

$$\begin{aligned}J_2 =&2\int_0^t u'(1,s)\Big(\int_0^s k(s-\tau)u(1,\tau)d\tau\Big)ds \\ &\le \frac{\varepsilon}{2}\widetilde{S}(t) + \frac{1}{\varepsilon}\|k\|_{L^1(0,T)}\int_0^t \Big(\int_s^t |k(s-\tau)|d\tau\Big)\widetilde{S}(s)ds,\end{aligned} \tag{3.6}$$

$$J_3 = 2\int_0^t \langle f(s), u'(s)\rangle ds \le \int_0^t \|f(s)\|ds + \int_0^t \|f(s)\|\widetilde{S}(s)ds. \tag{3.7}$$

By $(H_1), (\widetilde{H}_3), (\widetilde{H}_4)$, we deduce that

$$\widetilde{S}(0) - K\|u_0\|^2 + \frac{1}{\varepsilon}\int_0^t g^2(s)ds + \int_0^t \|f(s)\|ds \le \frac{C_2}{2}, \forall t \ge 0, \tag{3.8}$$

where C_2 is a positive constant independent of t.
Combining (3.4)-(3.8) and using $(\widetilde{H}_2)$, we obtain

$$\widetilde{S}(t) \le C_2 + 2\int_0^t \Big[\frac{1}{\varepsilon}\|k\|_{L^1(0,T)}\int_s^t |k(s-\tau)|d\tau + \|f(s)\|\Big]\widetilde{S}(s)ds. \tag{3.9}$$

From $(\widetilde{H}_3), (\widetilde{H}_4)$, we have that

$$\begin{aligned}&\int_0^t \Big(\frac{1}{\varepsilon}\|k\|_{L^1(0,T)}\int_s^t |k(s-\tau)|d\tau + \|f(s)\|\Big)ds \\ &\quad = \frac{1}{\varepsilon}\|k\|_{L^1(0,T)}\int_0^t \Big(\int_0^\tau |k(s-\tau)|ds\Big)d\tau + \int_0^t \|f(s)\|ds \\ &\quad \le C_3, \forall t \ge 0,\end{aligned} \tag{3.10}$$

where C_3 is a positive constant independent of t.
From (3.9),(3.10) and applying Gronwall's inequality, we obtain

$$\widetilde{S}(t) \leq C_2 \exp(C_3) \equiv C, \forall t \geq 0, \tag{3.11}$$

where C is a positive constant independent of t.
From (1.1),(3.4) and (3.11), we deduce that $\big(u(t), P(t)\big)$ defined on $\mathbb{R}_+$. And it follows that

$$\widetilde{S}(t) \geq \|u'(t)\|^2 + \|u_x(t)\|^2, \tag{3.12}$$

then

$$\|u'(t)\|^2 + \|u_x(t)\|^2 \leq C, \forall t \geq 0. \tag{3.13}$$

Hence, Lemma 4 is completely proved. □

Theorem 2. *Let* $(H_1), (\widehat{H}_2), (\widehat{H}_3) - (\widehat{H}_5)$ *hold. Then the component* $u(t)$ *in the weak solution* $\big(u(t), P(t)\big)$ *of problem* (1.1)-(1.5) *decays exponentially to zero as* $t \longrightarrow +\infty$ *in the following sense: there exists positive constant* N *such that*

$$\|u'(t)\|^2 + \|u_x(t)\|^2 \leq N e^{-\alpha t}, \forall t \geq 0. \tag{3.14}$$

Proof. We use the following functional

$$E(t) = E_0(t) + E_1(t), \tag{3.15}$$

where

$$\begin{cases} E_0(t) = \|u'(t)\|^2 + \|u_x(t)\|^2 + u^2(1,t), \\ E_1(t) = \langle u(t), u'(t) \rangle - \frac{1}{2} u^2(1,t). \end{cases} \tag{3.16}$$

Then we have some following estimates
Estimating $E(t)$.
From $(3.16)_2$, we deduce that

$$\begin{aligned} |E_1(t)| &\leq \|u_x(t)\| \|u'(t)\| + \frac{1}{2} u^2(1,t) \\ &\leq \frac{1}{2} \big(\|u'(t)\|^2 + \|u_x(t)\|^2 + u^2(1,t) \big). \end{aligned} \tag{3.17}$$

From (3.15)-(3.17), we obtain the following estimation

$$\frac{1}{2} E_0(t) \leq E(t) \leq \frac{3}{2} E_0(t). \tag{3.18}$$

Estimating $E'(t)$.
Differentiating (3.3) with respect to t, we get

$$\begin{aligned}
&E_0'(t) + 2\lambda\|u'(t)\|^2 + 2|u'(1,t)|^2 \\
&\qquad = -2g(t)u'(1,t) + 2u'(1,t)\int_0^t k(t-s)u(1,s)ds \\
&\qquad\quad - 2K\langle u(t), u'(t)\rangle + 2\langle f(t), u'(t)\rangle \\
&\qquad \le g^2(t) + \frac{3}{2}|u'(1,t)|^2 + 2\Big(\int_0^t k(t-s)u(1,s)ds\Big)^2 \\
&\qquad\quad + |K|\big(\|u_x(t)\|^2 + \|u'(t)\|^2\big) + \frac{1}{\varepsilon_1}\|f(t)\|^2 + \varepsilon_1\|u'(t)\|^2.
\end{aligned} \tag{3.19}$$

Then we receive from (3.19) after some rearrangements, that

$$\begin{aligned}
&E_0'(t) + (2\lambda - |K| - \varepsilon_1)\|u'(t)\|^2 - |K|\|u_x(t)\|^2 + \frac{1}{2}|u'(1,t)|^2 \\
&\qquad \le g^2(t) + 2C\Big(\int_0^t k(t-s)ds\Big)^2 + \frac{1}{\varepsilon_1}\|f(t)\|^2.
\end{aligned} \tag{3.20}$$

Now multiply the equation (1.1) by u, then integrate on $(0,1)$, we obtain

$$\begin{aligned}
&E_1'(t) - \|u'(t)\|^2 + \|u_x(t)\|^2 + u^2(1,t) \\
&\qquad = -g(t)u(1,t) + u(1,t)\int_0^t k(t-s)u(1,s)ds \\
&\qquad\quad - K\|u(t)\|^2 - \lambda\langle u'(t), u(t)\rangle + \langle f(t), u(t)\rangle \\
&\qquad \le g^2(t) + \frac{1}{2}u^2(1,t) + \Big(\int_0^t k(t-s)u(1,s)ds\Big)^2 \\
&\qquad\quad - K\|u(t)\|^2 + \frac{\lambda}{2}\big(\|u'(t)\|^2 + \|u(t)\|^2\big) \\
&\qquad\quad + \frac{1}{4\varepsilon_2}\|f(t)\|^2 + \varepsilon_2\|u(t)\|^2.
\end{aligned} \tag{3.21}$$

Hence,

$$\begin{aligned}
&E_1'(t) - \Big(1 + \frac{\lambda}{2}\Big)\|u'(t)\|^2 + \Big[1 - \Big(|K| + \frac{\lambda}{2} + \varepsilon_2\Big)\Big]\|u_x(t)\|^2 + \frac{1}{2}u^2(1,t) \\
&\qquad \le g^2(t) + C\Big(\int_0^t k(t-s)ds\Big)^2 + \frac{1}{4\varepsilon_2}\|f(t)\|^2.
\end{aligned} \tag{3.22}$$

Combine (3.20),(3.22), we deduce that

$$E'(t)+\left[\frac{3}{2}\lambda-(1+|K|+\varepsilon_1)\right]\|u'(t)\|^2+\left[1-\left(\frac{\lambda}{2}+2|K|+\varepsilon_2\right)\right]\|u_x(t)\|^2 + u^2(1,t) \leq 2g^2(t)+3C\left(\int_0^t k(t-s)ds\right)^2+\left(\frac{1}{\varepsilon_1}+\frac{1}{4\varepsilon_2}\right)\|f(t)\|^2. \tag{3.23}$$

Put

$$\gamma=\frac{2}{3}\min\left\{\frac{3}{2}\lambda-(1+|K|+\varepsilon_1), 1-\left(\frac{\lambda}{2}+2|K|+\varepsilon_2\right)\right\}, \tag{3.24}$$

By assumption $(\widehat{H}_2)$, we have $\gamma>0$. Then we get (3.23), that

$$E'(t)+\frac{3}{2}\gamma E_0(t)\leq 2g^2(t)+3C\left(\int_0^t k(t-s)ds\right)^2+\left(\frac{1}{\varepsilon_1}+\frac{1}{4\varepsilon_2}\right)\|f(t)\|^2. \tag{3.25}$$

From (3.18), (3.25), we deduce that

$$E'(t)+\gamma E(t)\leq 2g^2(t)+3C\left(\int_0^t k(t-s)ds\right)^2+\left(\frac{1}{\varepsilon_1}+\frac{1}{4\varepsilon_2}\right)\|f(t)\|^2. \tag{3.26}$$

Put $\eta=\min\left\{1,\frac{\gamma}{\alpha}\right\}$. Then integrating (3.26) with respect to t, we get

$$E(t)\leq\frac{1}{\eta}\left[\eta E(0)+2\int_0^t e^{\alpha s}g^2(s)ds+3C\int_0^t e^{\alpha s}\left(\int_0^s k(s-r)dr\right)^2 ds + \frac{\varepsilon_1+4\varepsilon_2}{4\varepsilon_1\varepsilon_2}\int_0^t e^{\alpha s}\|f(s)\|^2 ds\right]e^{-\alpha t}, \forall t\geq 0. \tag{3.27}$$

Using $(H_1),(\widetilde{H}_5)$, we conclude that

$$E(t)\leq \widetilde{N}e^{-\alpha t}, \forall t\geq 0, \tag{3.28}$$

where

$$\frac{1}{\eta}\left[\eta E(0)+2\int_0^{+\infty} e^{\alpha s}g^2(s)ds+3C\int_0^{+\infty} e^{\alpha s}\left(\int_0^s k(s-r)dr\right)^2 ds + \frac{\varepsilon_1+4\varepsilon_2}{4\varepsilon_1\varepsilon_2}\int_0^t e^{\alpha s}\|f(s)\|^2 ds\right]\leq \widetilde{N}. \tag{3.29}$$

From (3.18),(3.28), we have

$$E_0(t) \leq 2E(t) \leq 2\widetilde{N}e^{-\alpha t}, \forall t \geq 0. \tag{3.30}$$

Hence, by putting $N = 2\widetilde{N}$, we deduce from (3.30), that

$$\|u'(t)\|^2 + \|u_x(t)\|^2 \leq Ne^{-\alpha t}, \forall t \geq 0. \tag{3.31}$$

□

Now we will study the decays of both component $u(t)$ and $P(t)$ of solution $\big(u(t), P(t)\big)$ of problem (1.1)-(1.5). In order to investigate this, we modify some more assumptions as follows
$(\widehat{H}_1)$ $u_0 \in V \cap H^2, u_1 \in V,$
$(\widehat{H}_3)$ $f_t \in L^1(\mathbb{R}_+; L^2) \cap L^2(\mathbb{R}_+; L^2),$
$(\widehat{H}_4)$ $g' \in L^2(\mathbb{R}_+); k', h' \in L^1(\mathbb{R}_+), k(0) = 0,$
$(\widehat{H}_5)$ There exists a constant $\beta > 0$ such that

$$\begin{cases} \int_0^{+\infty} e^{\beta t}|g'(t)|^2 dt < +\infty, \\ \int_0^{+\infty} e^{\beta t}\left(\int_0^t k'(t-s)ds\right)^2 dt < +\infty, \\ \int_0^{+\infty} e^{\beta t}\|f_t(t)\|^2 dt < +\infty. \end{cases}$$

By Theorem 1 in [11], under assumptions $(\widehat{H}_1), (\widetilde{H}_2) - (\widetilde{H}_4), (\widehat{H}_3), (\widehat{H}_4)$ and for any $T > 0$, problem (1.1)-(1.5) has a unique weak solution (u, P) such that

$$\begin{cases} u \in L^\infty(0,T; V \cap H^2), u_t \in L^\infty(0,T; V), u_{tt} \in L^\infty(0,T; L^2), \\ u \in H^2(0,T), P \in H^1(0,T). \end{cases} \tag{3.32}$$

Then we receive the lemma as follows.

Lemma 5. *Let $(\widehat{H}_1), (\widetilde{H}_2) - (\widetilde{H}_4), (\widehat{H}_3), (\widehat{H}_4)$ hold. There is a unique solution $\big(u(t), P(t)\big)$ of problem (1.1)-(1.5) defined on $\mathbb{R}_+$.*
Moreover,

$$\begin{cases} \|u''(t)\|^2 + \|u_x'(t)\|^2 \leq \widehat{C}, \forall t \geq 0, \\ \|P\|_{L^\infty(\mathbb{R}_+)} \leq \overline{C}, \end{cases} \tag{3.33}$$

where $\widehat{C}, \overline{C}$ are positive constants independent of t.

Proof. From (3.4) and (3.11), we have

$$\|u'(t)\|^2+\|u_x(t)\|^2+u^2(1,t)+2\int_0^t |u'(1,s)|^2ds+2\lambda\int_0^t \|u'(s)\|^2ds \leq C, \forall t \geq 0. \tag{3.34}$$

Now, by differentiating equation (1.1) with respect to t, multiplying by u'', we receive after some rearrangements in which condition $k(0)=0$ is used, that

$$\begin{aligned} X(t) =& X(0) + K\|u'(t)\|^2 - K\|u'(0)\|^2 - 2\int_0^t g'(s)u''(1,s)ds \\ &+ 2\int_0^t u''(1,s)\Big(\int_0^s k'(s-\tau)u(1,\tau)d\tau\Big)ds \\ &+ 2\int_0^t \langle f_s(s), u''(s)\rangle ds \end{aligned} \tag{3.35}$$

where

$$\begin{aligned} X(t) =& \|u''(t)\|^2 + \|u_x'(t)\|^2 + |u'(1,t)|^2 + 2\int_0^t |u''(1,s)|^2ds \\ &+ 2\lambda\int_0^t \|u''(s)\|^2ds. \end{aligned} \tag{3.36}$$

By assumptions $(\widehat{H}_1), (\widetilde{H}_2), (\widehat{H}_3), (\widehat{H}_4)$, and the same method in Lemma 4, we obtain

$$X(t) \leq \widehat{C}, \forall t \geq 0, \tag{3.37}$$

where $\widehat{C}$ is a positive constant independent of t.
In addition, we also have

$$\begin{cases} |P(t)| \leq \widetilde{C}, \forall t \geq 0, \\ \int_0^t |P'(t)|^2dt \leq \widetilde{C}, \forall t \geq 0, \end{cases} \tag{3.38}$$

where $\widetilde{C}$ is a positive constant independent of t.
From (3.34),(3.36)-(3.38), we deduce that the solution $\big(u(t), P(t)\big)$ of problem (1.1)-(1.5) is defined on $\mathbb{R}_+$, and

$$\begin{cases} \|u''(t)\|^2 + \|u_x'(t)\|^2 \leq \widehat{C}, \forall t \geq 0, \\ \|P\|_{L^\infty(\mathbb{R}_+)} \leq \overline{C}. \end{cases} \tag{3.39}$$

□

Next we obtain the following theorem.

Theorem 3. *Let* $(\widehat{H}_1), (\widehat{H}_2), (\widetilde{H}_3), (\widetilde{H}_4), (\widehat{H}_3)-(\widehat{H}_5)$ *hold. The solution* $\big(u(t), P(t)\big)$ *of problem* (1.1)-(1.5) *decays exponentially to zero as* $t \to +\infty$ *in the following sense: there exist positive constant* $\widetilde{\widetilde{N}}, \overline{N}$ *and* $\overline{\gamma}$ *such that*

$$\begin{cases} \|u''(t)\|^2 + \|u_x'(t)\|^2 \le \widetilde{\widetilde{N}} e^{-\beta t}, \forall t \ge 0, \\ \|P\|_{L^\infty(\mathbb{R}_+)} \le \overline{N} e^{-\overline{\gamma} t}, \forall t \ge 0. \end{cases} \tag{3.40}$$

Proof. By using the following functional

$$\Gamma(t) = \Gamma_0(t) + \Gamma_1(t), \tag{3.41}$$

where

$$\begin{cases} \Gamma_0(t) = \|u''(t)\|^2 + \|u_x'(t)\|^2 + |u'(1,t)|^2, \\ \Gamma_1(t) = \langle u'(t), u''(t) \rangle - \frac{1}{2}|u'(1,t)|^2. \end{cases} \tag{3.42}$$

By using the same method as in Theorem 2, under assumptions mentioned in the theorem, we obtain $(3.40)_1$.
In addition, we have

$$\begin{aligned} |P(t)| &\le |u(1,t)| + |u'(1,t)| + \left| \int_0^t k(t-s) u(1,s) ds \right| + |g(t)| \\ &\le \|u_x(t)\| + \|u_x'(t)\| + C \left| \int_0^t k(t-s) ds \right| + |g(t)|. \end{aligned} \tag{3.43}$$

From (3.14), we deduce that

$$\begin{cases} \|u_x(t)\| \le \sqrt{N} e^{-\frac{\alpha}{2} t}, \forall t \ge 0, \\ \|u_x'(t)\| \le \sqrt{\widehat{N}} e^{-\frac{\beta}{2} t}, \forall t \ge 0. \end{cases} \tag{3.44}$$

And we also have from $(\widetilde{H}_5)$, that

$$\begin{cases} |g(t)| \le M_1 e^{-\frac{\alpha}{2} t}, \forall t \ge 0, \\ \left| \int_0^t k(t-s) ds \right| \le M_2 e^{-\frac{\alpha}{2} t}, \forall t \ge 0, \end{cases} \tag{3.45}$$

where M_1, M_2 are positive constants independent of t.
By putting $\overline{N} = \max\left\{\sqrt{N} + M_1 + M_2, \sqrt{\widehat{N}}\right\}$ and $\overline{\gamma} = \min\left\{\frac{\alpha}{2}, \frac{\beta}{2}\right\}$, we obtain

$$\|P\|_{L^\infty(\mathbb{R}_+)} \le \overline{N} e^{-\overline{\gamma} t}, \forall t \ge 0. \tag{3.46}$$

□

Remark. Studying the decay of the solution (u, P) in the case of both $K, \lambda \in \mathbb{R}$ is still an open problem.

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