The well-posedness of a semilinear wave equation associated with a linear integral equation at the boundary *†

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Abstract

In this paper, we prove the well-posedness for a mixed nonhomogeneous problem of a semilinear wave equation associated with a linear integral equation at the boundary.

1 Introduction

We investigate the following problem: Find a pair (u, Q) of functions satisfying

$$u_{tt} - \mu(t)u_{xx} + F(u, u_t) = f(x,t), 0 < x < 1, 0 < t < T, \tag{1.1}$$

$$u(0,t) = 0, \tag{1.2}$$

$$-\mu(t)u_x(1,t) = Q(t), \tag{1.3}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \tag{1.4}$$

where $F(u, u_t) = K|u|^{p-2}u + \lambda|u_t|^{q-2}u_t, p, q \geq 2$ with K, λ are given constants and u_0, u_1, f, μ are given functions satisfying conditions specified later, and the unknown function $u(x,t)$ and the unknown boundary value $Q(t)$ satisfy the following integral equation

$$Q(t) = K_1(t)u(1,t) + \lambda_1(t)u_t(1,t) - g(t) - \int_0^t k(t-s)u(1,s)ds, \tag{1.5}$$

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where g, k, K_1, λ_1 are given functions.

This problem is the mathematical model describing the shock of a rigid body and the viscoelastic bar (see [1], [2],[8],[9],[10],[11]) considered by several authors.

In [1], with $F(u, u_t) = Ku + \lambda u_t, \mu(t) \equiv a^2, f(x,t) = 0$, An, Trieu studied the equation $(1.1)_1$ in the domain $[0, l] \times [0, T]$ when the initial data is homogeneous, namely $u(x, 0) = u_t(x, 0) = 0$ and the boundary condition is given by

$$\begin{cases} Eu_x(0,t) = -f(t), \\ u(l,t) = 0, \end{cases} \tag{1.6}$$

where E is a constant.

In [6], Long, Dinh considered problem (1.1)-(1.4) with $\lambda_1(t) \equiv 0, K_1(t) = h \geq 0, \mu(t) = 1$, wherein the unknown function $u(x,t)$ and the unknown boundary value $Q(t)$ satisfying the following integral equation

$$Q(t) = hu(1,t) - g(t) - \int_0^t k(t-s)u(1,s)ds. \tag{1.7}$$

We note that Eq.(1.7) is deduced from a Cauchy problem for an ordinary differential equation at the boundary $x = 1$.

In [2] Bergounioux, Long and Dinh proved the unique existence for the solution of problem (1.1), (1.4) with $\mu(t) \equiv 1, F(u, u_t)$ is linear and the mixed boundary conditions (1.2), (1.3) standing for

$$u_x(0,t) = hu(0,t) + g(t) - \int_0^t k(t-s)u(0,s)ds, \tag{1.8}$$

$$u_x(1,t) + K_1 u(1,t) + \lambda_1 u_t(1,t) = 0, \tag{1.9}$$

In [13] Santos studied the asymptotic behavior of the solution of problem (1.1), (1.2), (1.4) in the case of $F(u, u_t) = 0$ associated with a boundary condition of memory type at $x = 1$ as follows

$$u(1,t) + \int_0^t g(t-s)\mu(s)u_x(1,s)ds = 0, \quad t > 0. \tag{1.10}$$

In [8], Long, Dinh and Diem obtained the unique existence, regularity and asymptotic expansion of the problem (1.1)-(1.4) in the case of $\mu(t) = 1, Q(t) = K_1 u(1,t) + \lambda_1 u_t(1,t), u_x(0,t) = P(t)$ where $P(t)$ satisfies (1.7) instead of $Q(t)$.

In $\big([9],[10],[11]\big)$, Long, Ut and Truc gave the unique existence, stability, regularity in time variable and asymptotic expansion for the solution of problem (1.1)-(1.5) when $F(u,u_t)=Ku+\lambda u_t$.

The present paper consists of two main parts. In Part 1 we prove a theorem of global existence and uniqueness of weak solution (u,Q) of problem (1.1) - (1.5). The proof is based on a Galerkin type approximation associated to various energy estimates-type bounds, weak-convergence and compactness arguments. The main difficulties encountered here are the boundary condition at $x=1$ and with the advent of the nonlinear term of $F(u,u_t)$. In order to solve these particular difficulties, stronger assumptions on the initial conditions u_0,u_1 and parameters K,λ will be modified. It's remarkable that the linearization method in the papers $\big([3],[7]\big)$ can't be used in $\big([2],[5],[6]\big)$. In the second part we show the stability of the solution of problem (1.1) - (1.5) in suitable spaces. The results obtained here may be considered as the generalizations of those in An and Trieu [1] and in Long, Dinh, Ut, Truc and Santos $\big([2],[3],[5]-[13]\big)$.

2 The existence and uniqueness of the solution

First we introduce some preliminary results and notations used in this paper. Put $\Omega=(0,1),Q_T=\Omega\times(0,T),T>0$. We omit the definitions of unsual fucntion space: $C^m(\overline{\Omega})$,$L^p=L^p(\Omega)$, $W^{m,p}(\Omega)$. We denote $W^{m,p}=W^{m,p}(\Omega)$, $L^p=W^{0,p}(\Omega)$, $H^m=W^{m,2}(\Omega)$, $1\le p\le\infty$, $m=0,1,...$

The norm in L^2 is denoted by $\|\cdot\|$. We also denote by $\langle\cdot,\cdot\rangle$ the scalar product in L^2 or pair of dual scalar product of a continuous linear functional with an element of a function space. We denote by $\|\cdot\|_X$ the norm of a Banach space X and by X' the dual space of X. We denote by $L^p(0,T;X)$, $1\le p\le\infty$ the Banach space of the real functions $u:(0,T)\to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)}=\left(\int_0^T\|u(t)\|_X^p dt\right)^{1/p}<\infty\quad\text{for}\quad 1\le p<\infty,$$

and

$$\|u\|_{L^\infty(0,T;X)}=\operatorname*{esssup}_{0<t<T}\|u(t)\|_X\qquad\text{for}\quad p=\infty.$$

Let $u(t),u'(t)=u_t(t),u''(t)=u_{tt}(t),u_x(t),u_{xx}(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

We put

$$V = \{v \in H^1 : v(0) = 0\}, \tag{2.1}$$

$$a(u,v) = \left\langle \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} \right\rangle = \int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} dx. \tag{2.2}$$

Here V is a closed subspace of H^1 and on V, $\|v\|_{H^1}$ and $\|v\|_V = \sqrt{a(v,v)}$ are two equivalent norms.
Then we have the following lemma.

Lemma 1. *The imbedding* $V \hookrightarrow C^0([0,1])$ *is compact and*

$$\|v\|_{C^0([0,1])} \le \|v\|_V, \tag{2.3}$$

for all $v \in V$.

We omit the detailed proof because of its certainty.
The process is continued by making the following essential assumptions:
(H_1) $K, \lambda \ge 0,$
(H_2) $u_0 \in V \cap H^2$, and $u_1 \in H^1$,
(H_3) $g, K_1, \lambda_1 \in H^1(0,T), \lambda_1(t) \ge \lambda_0 \ > 0, K_1(t) \ge 0,$
(H_4) $k \in H^1(0,T),$
(H_5) $\mu \in H^2(0,T), \mu(t) \ge \mu_0 > 0,$
(H_6) $f, f_t \in L^2(Q_T).$
Then we have the following theorem.

Theorem 1. *Let* (H_1)-(H_6) *hold. Then for every* $T > 0$, *there exists a unique weak solution* (u,Q) *of problem* (1.1)-(1.5) *such that*

$$\begin{cases} u \in L^\infty\big(0,T; V \cap H^2\big) \cap L^p(Q_T), \\ u_t \in L^\infty\big(0,T;V\big) \cap L^q(Q_T), u_{tt} \in L^\infty\big(0,T;L^2\big), \\ u(1,\cdot) \in H^2(0,T), Q \in H^1(0,T). \end{cases} \tag{2.4}$$

Remark 1. By $L^\infty(0,T;V) \subset L^p(Q_T) \quad \forall p, 1 \le p < \infty$, It follows from (2.4) that the component u in the weak solution (u,Q) of problem (1.1)-(1.5) satisfies

$$\begin{cases} u \in C^0\big(0,T;V\big) \cap C^1\big(0,T;L^2\big) \cap L^\infty\big(0,T;V \cap H^2\big), \\ u_t \in L^\infty(0,T;V). \end{cases} \tag{2.5}$$

Proof. The proof consists of Steps $1-4$.
Step 1. *The Galerkin approximation*: Let $\{\omega_j\}$ be a denumerable base of

$V \cap H^2$. We find the approximate solution of problem (1.1)-(1.5) in the form

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t)\omega_j, \tag{2.6}$$

where the coefficient functions c_{mj} satisfy the system of ordinary differential equations as follows

$$\begin{aligned} \langle u''_m(t), \omega_j\rangle + \mu(t)\langle u_{mx}(t), \omega_{jx}\rangle + Q_m(t)\omega_j(1) + \big\langle F\big(u_m(t), u'_m(t)\big), \omega_j\big\rangle& \\ = \langle f(t), \omega_j\rangle, 1 \leq j \leq m,& \end{aligned} \tag{2.7}$$

$$Q_m(t) = K_1(t)u_m(1,t) + \lambda_1(t)u'_m(1,t) - g(t) - \int_0^t k(t-s)u_m(1,s)ds, \tag{2.8}$$

$$\begin{cases} u_m(0) = u_{0m} = \displaystyle\sum_{j=1}^{m} \alpha_{mj}\omega_j \to u_0 & \text{strongly in} \quad V \cap H^2, \\ u'_m(0) = u_{1m} = \displaystyle\sum_{j=1}^{m} \beta_{mj}\omega_j \to u_1 & \text{strongly in} \quad H^1. \end{cases} \tag{2.9}$$

From the assumptions of Theorem 1, system (2.7)-(2.9) has solutions (u_m, Q_m) on some interval $[0, T_m]$. The following estimates allow one to take $T_m = T$ for all m.

Step 2. *A priori estimates*:

A priori estimates I. Substituting (2.8) into (2.7), then multiplying the j^{th} equation of (2.7) by $c'_{mj}(t)$ and summing up with respect to j, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u'_m(t)\|^2 + \frac{1}{2}\mu(t)\frac{d}{dt}\|u_{mx}(t)\|^2 + \Big[K_1(t)u_m(1,t) + \lambda_1(t)u'_m(1,t)& \\ - g(t) - \int_0^t k(t-s)u_m(1,s)ds\Big]u'_m(1,t) + \big\langle F\big(u_m, u'_m\big), u'_m(t)\big\rangle& \\ = \langle f(t), u'_m(t)\rangle.& \end{aligned} \tag{2.10}$$

Integrating (2.10) with respect to t, we get after some rearrangements

$$\begin{aligned} S_m(t) =& S_m(0) + \int_0^t \mu'(s)\|u_{mx}(s)\|^2 ds + \int_0^t K'_1(s)u_m^2(1,s)ds \\ &+ 2\int_0^t g(s)u'_m(1,s)ds + 2\int_0^t \langle f(s), u'_m(s)\rangle ds \\ &+ 2\int_0^t u'_m(1,s)\Big(\int_0^s k(s-\tau)u_m(1,\tau)d\tau\Big)ds, \end{aligned} \tag{2.11}$$

where

$$\begin{aligned}S_m(t) =&\|u_m'(t)\|^2 + \mu(t)\|u_{mx}(t)\|^2 + K_1(t)u_m^2(1,t)\\&+ \frac{2K}{p}\|u_m(t)\|_{L^p}^p + 2\lambda\int_0^t \|u_m'(s)\|_{L^q}^q ds\\&+ 2\int_0^t \lambda_1(s)|u_m'(1,s)|^2 ds.\end{aligned} \tag{2.12}$$

Using the inequality

$$2ab \le \beta a^2 + \frac{1}{\beta}b^2, \quad \forall a, b \in \mathbb{R}, \beta > 0, \tag{2.13}$$

and the following inequalities

$$S_m(t) \ge \|u_m'(t)\|^2 + \mu_0\|u_{mx}(t)\|^2 + 2\lambda_0\int_0^t |u_m'(1,s)|^2 ds, \tag{2.14}$$

$$|u_m(1,t)| \le \|u_m(t)\|_{C^0(\overline{\Omega})} \le \|u_{mx}(t)\| \le \sqrt{\frac{S_m(t)}{\mu_0}}, \tag{2.15}$$

we shall estimate respectively the following terms on the right-hand side of (2.11) as follows

$$\int_0^t \mu'(s)\|u_{mx}(s)\|^2 ds \le \frac{1}{\mu_0}\int_0^t |\mu'(s)|S_m(s)ds, \tag{2.16}$$

$$\int_0^t K_1'(s)u_m^2(1,s)ds \le \frac{1}{\mu_0}\int_0^t |K_1'(s)|S_m(s)ds, \tag{2.17}$$

$$2\int_0^t g(s)u_m'(1,s)ds \le \frac{1}{\beta}\|g\|_{L^2(0,T)}^2 + \frac{\beta}{2\lambda_0}S_m(t), \tag{2.18}$$

$$\begin{aligned}2\int_0^t u_m'(1,s)&\left(\int_0^s k(s-\tau)u_m(1,\tau)d\tau\right)ds\\&\le \frac{\beta}{2\lambda_0}S_m(t) + \frac{1}{\beta\mu_0}T\|k\|_{L^2(0,T)}^2\int_0^t S_m(s)ds,\end{aligned} \tag{2.19}$$

$$2\int_0^t \langle f(s), u_m'(s)\rangle ds \le \|f\|_{L^2(Q_T)}^2 + \int_0^t S_m(s)ds. \tag{2.20}$$

In addition, from the assumptions $(H_1),(H_2),(H_5)$ and the embedding $H^1 \hookrightarrow L^p(0,1),\quad p\geq 1$, there exists a positive constant C_1 such that

$$\begin{aligned} S_m(0)=&\|u_{1m}\|^2+\mu(0)\|u_{0mx}\|^2+K_1(0)u_{0m}^2(1) \\ &+\frac{2K}{p}\|u_{0m}\|_{L^p}^p \leq C_1 \text{ for all } m. \end{aligned} \tag{2.21}$$

Combining (2.11), (2.12), (2.16)-(2.21), we obtain

$$\begin{aligned} S_m(t) \leq & C_1+\frac{1}{\beta}\|g\|_{L^2(0,T)}^2+\|f\|_{L^2(Q_T)}^2+\frac{\beta}{\lambda_0}S_m(t) \\ &+\int_0^t\left[1+\frac{1}{\beta\mu_0}T\|k\|_{L^2(0,T)}^2+\frac{1}{\mu_0}\big(|\mu'(s)|+|K_1'(s)|\big)\right]S_m(s)ds. \end{aligned} \tag{2.22}$$

By choosing $\beta=\frac{\lambda_0}{2}$, it follows from (2.22), that

$$S_m(t)\leq M_T^{(1)}+\int_0^t N_T^{(1)}(s)S_m(s)ds, \tag{2.23}$$

where

$$\begin{cases} M_T^{(1)}=2C_1+\frac{4}{\lambda_0}\|g\|_{L^2(0,T)}^2+2\|f\|_{L^2(Q_T)}^2, \\ N_T^{(1)}(s)=2\left[1+\frac{2}{\lambda_0\mu_0}T\|k\|_{L^2(0,T)}^2+\frac{1}{\mu_0}\big(|\mu'(s)|+|K_1'(s)|\big)\right], \\ N_T^{(1)}\in L^1(0,T). \end{cases} \tag{2.24}$$

By Gronwall's lemma, we deduce from (2.23), (2.24), that

$$S_m(t)\leq M_T^{(1)}exp\left(\int_0^t N_T^{(1)}(s)ds\right)\leq C_T, \text{ for all } t\in[0,T]. \tag{2.25}$$

A priori estimates II. Now differentiating (2.7) with respect to t we have

$$\begin{aligned} &\langle u_m'''(t),\omega_j\rangle+\mu(t)\langle u_{mx}'(t),\omega_{jx}\rangle+\mu'(t)\langle u_{mx}(t),\omega_{jx}\rangle+Q_m'(t)\omega_j(1) \\ &+\big\langle K(p-1)|u_m|^{p-2}u_m'+\lambda(q-1)|u_m'|^{q-2}u_m'',\omega_j\big\rangle=\langle f'(t),\omega_j\rangle, \end{aligned} \tag{2.26}$$

for all $1\leq j\leq m$.
Multiplying the j^{th} equation of (2.28) by $c_{mj}''(t)$, summing up with respect to j and then integrating with respect to the time variable from 0 to t, we have

after some persistent rearrangements

$$\begin{aligned}
X_m(t) =& X_m(0) + 2\mu'(0)\langle u_{0mx}, u_{1mx}\rangle - 2\mu'(t)\langle u_{mx}(t), u'_{mx}(t)\rangle \\
&+ 3\int_0^t \mu'(s)\|u'_{mx}(s)\|^2 ds + 2\int_0^t \mu''(s)\langle u_{mx}(s), u'_{mx}(s)\rangle ds \\
&- 2\int_0^t \big[K_1'(s) - k(0)\big] u_m(1,s) u''_m(1,s) ds \\
&- 2\int_0^t \big[K_1(s) + \lambda_1'(s)\big] u'_m(1,s) u''_m(1,s) ds \\
&+ 2\int_0^t u''_m(1,s)\left(g'(s) + \int_0^s k'(s-\tau) u_m(1,\tau) d\tau\right) ds \\
&- 2\int_0^t \big\langle K(p-1)|u_m(s)|^{p-2} u'_m(s), u''_m(s)\big\rangle ds \\
&+ 2\int_0^t \langle f'(s), u''_m(s)\rangle ds,
\end{aligned} \tag{2.27}$$

where

$$\begin{aligned}
X_m(t) =& \|u''_m(t)\|^2 + \mu(t)\|u'_{mx}(t)\|^2 + 2\int_0^t \lambda_1(s)|u''_m(1,s)|^2 ds \\
&+ \frac{8}{q^2}(q-1)\lambda \int_0^t \left\| \frac{\partial}{\partial t}\big(|u'_m(s)|^{\frac{q-2}{2}} u'_m(s)\big)\right\|^2 ds.
\end{aligned} \tag{2.28}$$

From the assumptions $(H_1), (H_2), (H_5), (H_6)$ and the embedding $H^1(0,1) \hookrightarrow L^p(0,1)$, $p \geq 1$, there exist positive constants D_1, D_2 depending on $\mu(0), u_0, u_1, K, \lambda, f$ such that

$$\begin{cases}
X_m(0) = \|u''_m(0)\|^2 + \mu(0)\|u_{1mx}\|^2 \\
\qquad\qquad \leq \mu(0)\|u_{0mxx}\| + K\|u_{0m}\|^{p-1}_{L^{2p-2}} \\
\qquad\qquad\quad + \lambda\|u_{1m}\|^{q-1}_{L^{2q-2}} + \|f(0)\| + \mu(0)\|u_{1mx}\|^2 \leq D_1, \\
2\mu'(0)\langle u_{0mx}, u_{1mx}\rangle \leq 2|\mu'(0)|\|u_{0mx}\|\|u_{1mx}\| \leq D_2,
\end{cases} \tag{2.29}$$

for all m.

Taking into account of inequality (2.13) where β is replaced by β_1 and the following inequalities

$$X_m(t) \geq \|u_m''(t)\|^2 + \mu_0 \|u_{mx}'(t)\|^2 + 2\lambda_0 \int_0^t |u_m''(1,s)|^2 ds, \tag{2.30}$$

$$|u_m(1,t)| \leq \|u_m(t)\|_{C^0(\overline{\Omega})} \leq \|u_{mx}(t)\| \leq \sqrt{\frac{S_m(t)}{\mu_0}} \leq \sqrt{\frac{C_T}{\mu_0}}, \tag{2.31}$$

$$|u_m'(1,t)| \leq \|u_m'(t)\|_{C^0(\overline{\Omega})} \leq \|u_{mx}'(t)\| \leq \sqrt{\frac{X_m(t)}{\mu_0}}, \tag{2.32}$$

we estimate, without any difficulties, the following terms in the right-hand side of (2.27) as follows

$$-2\mu'(t)\langle u_{mx}(t), u_{mx}'(t)\rangle \leq \beta_1 X_m(t) + \frac{1}{\beta_1 \mu_0^2} C_T |\mu'(t)|^2, \tag{2.33}$$

$$2\int_0^t \mu''(s)\langle u_{mx}(s), u_{mx}'(s)\rangle ds \leq \frac{C_T}{\beta_1 \mu_0^2}\|\mu''\|_{L^2(0,T)}^2 + \beta_1 \int_0^t X_m(s) ds, \tag{2.34}$$

$$3\int_0^t \mu'(s)\|u_{mx}'(s)\|^2 ds \leq \frac{3}{\mu_0}\int_0^t |\mu'(s)| X_m(s) ds, \tag{2.35}$$

$$\begin{aligned} -2\int_0^t &\left[K_1'(s) - k(0)\right] u_m(1,s) u_m''(1,s) ds \\ &\leq \frac{C_T}{\mu_0 \beta_1}\|K_1' - k(0)\|_{L^2(0,T)}^2 + \frac{\beta_1}{2\lambda_0} X_m(t), \end{aligned} \tag{2.36}$$

$$\begin{aligned} -2\int_0^t &\left[K_1(s) + \lambda_1'(s)\right] u_m'(1,s) u_m''(1,s) ds \\ &\leq \frac{2}{\mu_0 \beta_1}\int_0^t \left[K_1^2(s) + |\lambda_1'(s)|^2\right] X_m(s) ds + \frac{\beta_1}{2\lambda_0} X_m(t), \end{aligned} \tag{2.37}$$

$$\begin{aligned} 2\int_0^t & u_m''(1,s)\left(g'(s) + \int_0^s k'(s-\tau) u_m(1,\tau) d\tau\right) ds \\ &\leq \frac{\beta_1}{2\lambda_0} X_m(t) + \frac{2}{\beta_1}\left[\|g'\|_{L^2(0,T)}^2 + \frac{C_T}{\mu_0} T \|k'\|_{L^1(0,T)}^2\right], \end{aligned} \tag{2.38}$$

$$\begin{aligned}&-2K(p-1)\int_0^t \langle |u_m(s)|^{p-2}u_m'(s), u_m''(s)\rangle ds \\ &\qquad\qquad \le 2\frac{p-1}{\sqrt{\mu_0}}K\left(\frac{C_T}{\mu_0}\right)^{\frac{p-2}{2}}\int_0^t X_m(s)ds,\end{aligned} \tag{2.39}$$

$$2\int_0^t \langle f'(s), u_m''(s)\rangle ds \le \beta_1\int_0^t X_m(s)ds + \frac{1}{\beta_1}\|f'\|^2_{L^2(Q_T)}. \tag{2.40}$$

In terms of (2.27), (2.29), (2.33)-(2.40), we abtain that

$$\begin{aligned}X_m(t) \le & D_1 + D_2 + \frac{C_T}{\beta_1\mu_0^2}|\mu'(t)|^2 + \frac{C_T}{\beta_1\mu_0^2}\|\mu''\|^2_{L^2(0,T)} \\ & + \frac{C_T}{\beta_1\mu_0}\|K_1' - k(0)\|^2_{L^2(0,T)} + \frac{1}{\beta_1}\|f'\|^2_{L^2(Q_T)} \\ & + \beta_1\left(1+\frac{1}{2\lambda_0}\right)X_m(t) + \frac{2}{\beta_1}\left[\|g'\|^2_{L^2(0,T)} + \frac{C_T}{\mu_0}T\|k'\|^2_{L^1(0,T)}\right] \\ & + 2\int_0^t\left[\beta_1 + \frac{3}{2\mu_0}|\mu'(s)| + \frac{1}{\beta_1\mu_0}\left(K_1^2(s) + |\lambda_1'(s)|^2\right)\right. \\ & \qquad\qquad \left. + \frac{p-1}{\sqrt{\mu_0}}K\left(\frac{C_T}{\mu_0}\right)^{\frac{p-2}{2}}\right]\int_0^t X_m(s)ds,\end{aligned} \tag{2.41}$$

By the choice of $\beta_1 > 0$ such that

$$\beta_1\left(1+\frac{3}{2\lambda_0}\right) \le \frac{1}{2}, \tag{2.42}$$

we receive

$$X_m(t) \le \widetilde{M}_T^{(2)}(t) + \int_0^t N_T^{(2)}(s)X_m(s)ds, \tag{2.43}$$

where

$$\begin{cases}\widetilde{M}_T^{(2)}(t) = 2D_1 + 2D_2 + \frac{2C_T}{\beta_1\mu_0^2}|\mu'(t)|^2 + \frac{2C_T}{\beta_1\mu_0^2}\|\mu''\|^2_{L^2(0,T)} \\ \qquad\qquad + \frac{2C_T}{\beta_1\mu_0}\|K_1' - k(0)\|^2_{L^2(0,T)} + \frac{2}{\beta_1}\|f'\|^2_{L^2(Q_T)} \\ \qquad\qquad + \frac{4}{\beta_1}\left[\|g'\|^2_{L^2(0,T)} + \frac{C_T}{\mu_0}T\|k'\|^2_{L^1(0,T)}\right], \\ N_T^{(2)}(s) = 4\left[\beta_1 + \frac{3}{2\mu_0}|\mu'(s)| + \frac{1}{\beta_1\mu_0}\left(K_1^2(s) + |\lambda_1'(s)|^2\right)\right. \\ \qquad\qquad \left. + \frac{p-1}{\sqrt{\mu_0}}K\left(\frac{C_T}{\mu_0}\right)^{\frac{p-2}{2}}\right], \\ N_T^{(2)} \in L^1(0,T).\end{cases} \tag{2.44}$$

From the assumptions $(H_3)-(H_6)$ and the embedding $H^1(0,T) \hookrightarrow C^0([0,T])$, we deduce that

$$\widetilde{M}_T^{(2)}(t) \leq M_T^{(2)} \quad \text{for all } t \in [0,T], \tag{2.45}$$

where $M_T^{(2)}$ is a positive constant depeding on $T, D_1, D_2, C_T, \mu, \beta_1, g, f, K_1, \lambda_1$. From (2.43)-(2.45) and Gronwall's inequality, we derive that

$$X_m(t) \leq M_T^{(2)} exp\bigg(\int_0^t N_T^{(2)}(s)ds\bigg) < D_T \quad \text{for all } t \in [0,T]. \tag{2.46}$$

On the other hand, we deduce from (2.8), (2.12), (2.25), (2.28), (2.46), that

$$\begin{aligned} \|Q'_m\|^2_{L^2(0,T)} \leq & \frac{5D_T}{2\lambda_0}\|\lambda_1\|^2_\infty + \frac{5T^2C_T}{\mu_0}\|k'\|^2_{L^2(0,T)} + 5\|g'\|^2_{L^2(0,T)} \\ & + \frac{5D_T}{\mu_0}\bigg(\|K_1 + \lambda'_1\|^2_{L^2(0,T)}\|K'_1 - k(0)\|^2_{L^2(0,T)}\bigg), \end{aligned} \tag{2.47}$$

where $\|\lambda_1\|_\infty = \|\lambda_1\|_{L^\infty(0,T)}$.
Taking into account assumptions $(H_3), (H_4)$, we deduce from (2.47), that

$$\|Q_m\|_{H^1(0,T)} \leq C_T \quad \text{for all } m, \tag{2.48}$$

where C_T is a posistive constant depending only on T.
Step 3. Limiting process. In the main results of (2.12), (2.25), (2.28),(2.46) and (2.48), we conclude the existence of a subsequence of (u_m, Q_m) still also so denoted, such that

$$\begin{cases} u_m \to u & \text{in} & L^\infty(0,T;V) & \text{weak}^\star, \\ u_m \to u & \text{in} & L^\infty(0,T;L^p) & \text{weak}^\star, \\ u'_m \to u' & \text{in} & L^\infty(0,T;V) & \text{weak}^\star, \\ u'_m \to u' & \text{in} & L^\infty(0,T;L^q) & \text{weak}^\star, \\ u''_m \to u'' & \text{in} & L^\infty(0,T;L^2) & \text{weak}^\star, \\ u_m(1,\cdot) \to u(1,\cdot) & \text{in} & H^2(0,T) & \text{weakly}, \\ |u_m|^{p-2}u_m \to \chi_1 & \text{in} & L^\infty(0,T;L^{p/p-1}) & \text{weak}^\star, \\ |u'_m|^{q-2}u'_m \to \chi_2 & \text{in} & L^\infty(0,T;L^{q/q-1}) & \text{weak}^\star, \\ Q_m \to \widetilde{Q} & \text{in} & H^1(0,T) & \text{weakly}. \end{cases} \tag{2.49}$$

With the help of the compactness lemma of J.L. Lions $\big([4] : p.57\big)$ and the embeddings $H^2(0,T) \hookrightarrow H^1(0,T), H^1(0,T) \hookrightarrow C^0([0,T])$, we can deduce from (2.49)$_{1,3,6,7}$ the existence of a subsequence still denoted by (u_m, Q_m) such that

$$\begin{cases} u_m \to u & \text{strongly in} \quad L^2(Q_T), \\ u'_m \to u' & \text{strongly in} \quad L^2(Q_T), \\ u_m(1,\cdot) \to u(1,\cdot) & \text{strongly in} \quad H^1(0,T), \\ u'_m(1,\cdot) \to u'(1,\cdot) & \text{strongly in} \quad C^0[0,T], \\ Q_m \to \widetilde{Q} & \text{strongly in} \quad C^0[0,T]. \end{cases} \tag{2.50}$$

The remarkable results of (2.8) and (2.50)$_{3-4}$ help us to affirm that

$$\begin{aligned} Q_m(t) \to & K_1(t)u(1,t) + \lambda_1(t)u'(1,t) - g(t) - \int_0^t k(t-s)u(1,s)ds \\ & \equiv Q(t) \quad \text{strongly in } C^0[0,T]. \end{aligned} \tag{2.51}$$

On account of (2.50)$_5$ and (2.51), we're enable to conclude that

$$Q(t) = \widetilde{Q}(t). \tag{2.52}$$

By means of the following inequality

$$||x|^\alpha x - |y|^\alpha y| \leq (\alpha+1)R^\alpha |x-y|, \forall x, y \in [-R, R] \text{ for all } R > 0, \alpha \geq 0, \tag{2.53}$$

it follows from (2.31), that

$$\left||u_m|^{p-2}u_m - |u|^{p-2}u\right| \leq (p-1)R^{p-2}|u_m - u|, R = \sqrt{\frac{C_T}{\mu_0}}. \tag{2.54}$$

Hence, it follows from (2.54), (2.50)$_1$, that

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \quad \text{strongly in} \quad L^2(Q_T). \tag{2.55}$$

By the same way, we're able to get from (2.53), with $R = \sqrt{\frac{D_T}{\mu_0}}$ and (2.49)$_3$, (2.50)$_2$, that

$$|u'_m|^{p-2}u'_m \to |u_t|^{p-2}u_t \quad \text{strongly in} \quad L^2(Q_T). \tag{2.56}$$

As a result, we deduce from (2.55), (2.56), that

$$F(u_m, u'_m) \to F(u, u_t) \quad \text{strongly in} \quad L^2(Q_T). \tag{2.57}$$

Passing to the limit in (2.7) − (2.9) by $(2.49)_{1,5}$, (2.51)-(2.52), (2.57), we have (u,Q) satisfying the problem

$$\begin{aligned}
&\langle u''(t),v\rangle+\mu(t)\langle u_x(t),v_x\rangle+Q(t)v(1)\\
&\qquad+\langle K|u(t)|^{p-2}u(t)+\lambda|u_t(t)|^{q-2}u_t(t),v\rangle\\
&\qquad\qquad=\langle f(t),v\rangle,\forall v\in V,
\end{aligned}\tag{2.58}$$

$$u(0)=u_0,u'(0)=u_1,\tag{2.59}$$

$$Q(t)=K_1(t)u(1,t)+\lambda_1(t)u_t(1,t)-g(t)-\int_0^t k(t-s)u(1,s)ds,\tag{2.60}$$

in $L^2(0,T)$ weakly. Nevertheless, we obtain from $(2.42)_5$, (2.57) and assumptions $(H_5)-(H_6)$, that

$$u_{xx}=\frac{1}{\mu(t)}\big[u''+F(u,u_t)-f\big]\in L^\infty\big(0,T;L^2\big).\tag{2.61}$$

Thus $u\in L^\infty\big(0,T;V\cap H^2\big)$ and the existence of the theorem is proved completely.
Step 4. Uniqueness of the solution.
We start this part by letting (u_1,Q_1) and (u_2,Q_2) be two weak solutions of problem (1.1)-(1.5) such that

$$\begin{cases}u_i\in L^\infty\big(0,T;V\cap H^2\big)\cap L^p(Q_T),\\ u_i'\in L^\infty\big(0,T;V\big)\cap L^q(Q_T),u_i''\in L^\infty\big(0,T;L^2\big),\\ u_i(1,\cdot)\in H^2(0,T),Q_i\in H^1(0,T)\quad i=1,2.\end{cases}\tag{2.62}$$

As a result, (u,Q) which $u=u_1-u_2$, and $Q=Q_1-Q_2$ satisfy the following variational problem

$$\begin{cases}\langle u''(t),v\rangle+\mu(t)\langle u_x(t),v_x\rangle+Q(t)v(1)\\ \qquad+K\big\langle|u_1(t)|^{p-2}u_1(t)-|u_2(t)|^{p-2}u_2(t),v\big\rangle\\ \qquad+\lambda\big\langle|u_1'(t)|^{q-2}u_1'(t)-|u_2'(t)|^{q-2}u_2'(t),v\big\rangle=0\quad\forall v\in V,\\ u(0)=u'(0)=0,\end{cases}\tag{2.63}$$

and

$$Q(t)=K_1(t)u(1,t)+\lambda_1(t)u'(1,t)-\int_0^t k(t-s)u(1,s)ds.\tag{2.64}$$

In the process of choosing $v = u'$ in (2.63)$_1$, and integrating with respect to t, it results in

$$\begin{aligned} S(t) \leq & \int_0^t \mu'(s)\|u_x(s)\|^2 ds + \int_0^t K_1'(s)u^2(1,s)ds \\ & + 2\int_0^t u'(1,s)\Big(\int_0^s k(s-\tau)u(1,\tau)d\tau\Big)ds \\ & - 2K\int_0^t \big\langle |u_1(s)|^{p-2}u_1(s) - |u_2(s)|^{p-2}u_2(s), u'(s)\big\rangle ds, \end{aligned} \tag{2.65}$$

where

$$\begin{aligned} S(t) =& \|u'(t)\|^2 + \mu(t)\|u_x(t)\|^2 + K_1(t)u^2(1,t) \\ & + 2\int_0^t \lambda_1(s)|u'(1,s)|^2 ds \end{aligned} \tag{2.66}$$

Noting that

$$S(t) \geq \|u'(t)\|^2 + \mu_0\|u_x(t)\|^2 + 2\lambda_0 \int_0^t |u'(1,s)|^2 ds, \tag{2.67}$$

$$|u(1,t)| \leq \|u(t)\|_{C^0(\overline{\Omega})} \leq \|u_x(t)\| \leq \sqrt{\frac{S(t)}{\mu(t)}} \leq \sqrt{\frac{S(t)}{\mu_0}}. \tag{2.68}$$

We again use inequalities (2.13) and (2.53) with $\alpha = p-2$, $R = max_{i=1,2}$ $\|u_i\|_{L^\infty(0,T;V)}$, then, it follows from (2.65)-(2.68), that

$$\begin{aligned} S(t) \leq & \frac{1}{\mu_0}\int_0^t \big(\|\mu'\|_\infty + |K_1'(s)|\big)S(s)ds + \frac{\beta}{2\lambda_0}S(t) \\ & + \frac{T}{\beta\mu_0}\|k\|^2_{L^2(0,T)}\int_0^t S(\tau)d\tau + \frac{1}{\sqrt{\mu_0}}(p-1)KR^{p-2}\int_0^t S(s)ds. \end{aligned} \tag{2.69}$$

Choosing $\beta > 0$, such that $\beta\frac{1}{2\lambda_0} \leq \frac{1}{2}$, we obtain from (2.69), that

$$S(t) \leq \int_0^t q_1(s)S(s)ds, \tag{2.70}$$

where

$$\begin{cases} q_1(s) = \frac{1}{\mu_0}\big(\|\mu'\|_\infty + |K_1'(s)|\big) + \frac{2T}{\beta\mu_0}\|k\|^2_{L^2(0,T)} + \frac{2}{\sqrt{\mu_0}}(p-1)KR^{p-2}, \\ q_1 \in L^2(0,T). \end{cases} \tag{2.71}$$

By Gronwall's lemma, we deduce that $S \equiv 0$ and Theorem 1 is proved completely. □

Remark 2. In the case of $p, q > 2$ and $K, \lambda < 0$, the question about the existence for the solutions of problem (1.1)-(1.5) is still open. However we have received the answer when $p = q = 2$ and $K, \lambda \in \mathbb{R}$ published in [11].

3 The stability of the solution

In this section, we assume that the functions u_0, u_1 satisfy (H_2). By Theorem 1, the problem (1.1)-(1.5) has a unique weak solution (u, Q) depending on $\mu, K, \lambda, f, K_1, \lambda_1, g, k$. So we have

$$u = u(\mu, K, \lambda, f, K_1, \lambda_1, g, k), Q = Q(\mu, K, \lambda, f, K_1, \lambda_1, g, k), \tag{3.1}$$

where $(\mu, K, \lambda, f, K_1, \lambda_1, g, k)$ satisfy the assumptions $(H_1), (H_3) - (H_6)$ and u_0, u_1 are fixed functions satisfying (H_2).
We put

$$\begin{aligned}\Im(\mu_0, \lambda_0) = \big\{(\mu, K, \lambda, f, K_1, \lambda_1, g, k) : (\mu, K, \lambda, f, K_1, \lambda_1, g, k) \\ \text{satisfy the assumptions} \quad (H_1), (H_3) - (H_6)\big\},\end{aligned}$$

where $\mu_0 > 0, \lambda_0 > 0$ are given constants.
Then there is a theorem as follows.

Theorem 2. *For every $T > 0$, let $(H_1) - (H_6)$ hold. Then the solutions of problem (1.1)-(1.5) are stable with respect to the data $(\mu, K, \lambda, f, K_1, \lambda_1, g, k)$, i.e., if $(\mu, K, \lambda, f, K_1, \lambda_1, g, k)$, $(\mu^j, K^j, \lambda^j, f^j, K_1^j, \lambda_1^j, g^j, k^j) \in \Im\,(\mu_0,\lambda_0)$, such that*

$$\begin{cases} \|\mu^j - \mu\|_{H^2(0,T)} \to 0, |K^j - K| + |\lambda^j - \lambda| \to 0, \\ \|f^j - f\|_{L^2(Q_T)} + \|f_t^j - f_t\|_{L^2(Q_T)} \to 0, \\ \|K_1^j - K_1\|_{H^1(0,T)} \to 0, \|\lambda_1^j - \lambda_1\|_{H^1(0,T)} \to 0, \\ \|g^j - g\|_{H^1(0,T)} \to 0, \|k^j - k\|_{H^1(0,T)} \to 0, \end{cases} \tag{3.2}$$

as $j \to +\infty$, then

$$\big(u_j, u_j', u_j(1, \cdot), Q_j\big) \to \big(u, u', u(1, \cdot), Q\big) \tag{3.3}$$

in $L^\infty(0, T; V) \times L^\infty(0, T; L^2) \times H^1(0, T) \times L^2(0, T)$ strongly as $j \to +\infty$ where $u_j = u\big(\mu^j, K^j, \lambda^j, f^j, K_1^j, \lambda_1^j, g^j, k^j\big)$, $Q_j = Q\big(\mu^j, K^j, \lambda^j, f^j, K_1^j, \lambda_1^j, g^j, k^j\big)$.

Proof. First of all, we have that if the data $\big(\mu, K, \lambda, f, K_1, \lambda_1, g, k\big)$ satisfy

$$\begin{cases} \|\mu\|_{H^2(0,T)} \leq \mu^\star, \quad 0 \leq K \leq K^\star, \quad 0 \leq \lambda \leq \lambda^\star, \\ \|f\|_{L^2(Q_T)} + \|f_t\|_{L^2(Q_T)} \leq f^\star, \\ \|K_1\|_{H^1(0,T)} \leq K_1^\star, \quad \|\lambda_1\|_{H^1(0,T)} \leq \lambda_1^\star, \\ \|g\|_{H^1(0,T)} \leq g^\star, \quad \|k\|_{H^1(0,T)} \leq k^\star, \end{cases} \tag{3.4}$$

where $\mu^\star, K^\star, \lambda^\star, f^\star, K_1^\star, \lambda_1^\star, g^\star, k^\star$ are fixed positive constants. Therefore the a priori estimates of the sequences $\{u_m\}$ and $\{Q_m\}$ in the proof of Theorem 1 satisfy

$$\|u_m'(t)\|^2 + \mu_0\|u_{mx}(t)\|^2 + 2\lambda_0\int_0^t |u_m'(1,s)|^2 ds \leq M_T, \forall t \in [0,T], \tag{3.5}$$

$$\|u_m''(t)\|^2 + \mu_0\|u_{mx}'(t)\|^2 + 2\lambda_0\int_0^t |u_m''(1,s)|^2 ds \leq M_T, \forall t \in [0,T], \tag{3.6}$$

$$\|Q_m\|_{H^1(0,T)} \leq M_T, \tag{3.7}$$

where M_T is a positive constant depending on $T, u_0, u_1, \mu_0, \lambda_0, \mu^\star, K^\star, \lambda^\star, f^\star$, (independent of $\mu, K, \lambda, f, K_1, \lambda_1, g, k$).
Hence the limit (u, Q) in suitable spaces of the sequence $\{(u_m, Q_m)\}$ defined by (2.6)-(2.8) is a weak solution of the problem (1.1)-(1.5) satisfying the estimates (3.5)-(3.7).
Now by (3.2) we can assume that there exist positive constants $\mu^\star, K^\star, \lambda^\star, f^\star$, $K_1^\star$, $\lambda_1^\star, g^\star, k^\star$ such that the data $\big(\mu^j, K^j, \lambda^j, f^j, K_1^j, \lambda_1^j, g^j, k^j\big)$ satisfy (3.4) with $\big(\mu, K, \lambda, f, K_1, \lambda_1, g, k\big) = \big(\mu^j, K^j, \lambda^j, f^j, K_1^j, \lambda_1^j, g^j, k^j\big)$. Then, by the above remark, we have that the solution (u_j, Q_j) of problem (1.1)-(1.5) corresonding to $\big(\mu, K, \lambda, f, K_1, \lambda_1, g, k\big) = \big(\mu^j, K^j, \lambda^j, f^j, K_1^j, \lambda_1^j, g^j, k^j\big)$ satisfy

$$\|u_j'(t)\|^2 + \mu_0\|u_{jx}(t)\|^2 + 2\lambda_0\int_0^t |u_j'(1,s)|^2 ds \leq M_T, \forall t \in [0,T], \tag{3.8}$$

$$\|u_j''(t)\|^2 + \mu_0\|u_{jx}'(t)\|^2 + 2\lambda_0\int_0^t |u_j''(1,s)|^2 ds \leq M_T, \forall t \in [0,T], \tag{3.9}$$

$$\|Q_j\|_{H^1(0,T)} \leq M_T. \tag{3.10}$$

Put

$$\begin{cases} \widetilde{\mu}_j = \mu^j - \mu, \quad \widetilde{K}_j = K^j - K, \quad \widetilde{\lambda}_j = \lambda^j - \lambda, \\ \widetilde{f}_j = f^j - f, \quad \widetilde{K_{1j}} = K_1^j - K_1, \quad \widetilde{\lambda_1}^j = \lambda_1^j - \lambda_1, \\ \widetilde{g}_j = g^j - g, \quad \widetilde{k}_j = k^j - k. \end{cases} \tag{3.11}$$

Consequently, $v_j = u_j - u, P_j = Q_j - Q$ satisfy the following variational problem

$$\begin{cases} \langle v_j''(t), v\rangle + \mu(t)\langle v_{jx}(t), v_x\rangle + P_j(t)v(1) \\ \qquad + K_j\big\langle |u_j|^{p-2}u_j - |u|^{p-2}u, v\big\rangle + \lambda_j\big\langle |u_j'|^{q-2}u_j' - |u'|^{q-2}u', v\big\rangle \\ \quad = \langle \widetilde{f}_j, v\rangle - \widetilde{\mu}_j(t)\langle u_{jx}(t), v_x\rangle - \widetilde{K_j}\langle |u|^{p-2}u, v\rangle - \widetilde{\lambda}_j\langle |u'|^{q-2}u', v\rangle \quad \forall v \in V, \\ v_j(0) = v_j'(0) = 0, \end{cases} \tag{3.12}$$

where

$$\begin{aligned} P_j(t) &= Q_j(t) - Q(t) \\ &= K_1(t)v_j(1,t) + \lambda_1(t)v_{jt}(1,t) - \int_0^t k(t-s)v_j(1,s)ds - \widehat{g}_j(t), \end{aligned} \tag{3.13}$$

$$\widehat{g}_j(t) = \widetilde{g}_j(t) - \widetilde{K_{1j}}(t)u_j(1,t) - \widetilde{\lambda}_{1j}(t)u_{jt}(1,t) + \int_0^t \widetilde{k}_j(t-s)u_j(1,s)ds. \tag{3.14}$$

Substituting $P_j(t)$ into (3.12), then taking $v = v_j'$ in $(3.12)_1$, afterwards integrating in t, we obtain

$$\begin{aligned} S_j(t) \leq &\int_0^t \mu_j'(s)\|v_{jx}(x)\|^2 ds + \int_0^t K_1'(s)v_j^2(1,s)ds \\ &+ 2\int_0^t v_j'(1,\tau)d\tau \int_0^\tau k(\tau-s)v_j(1,s)ds + 2\int_0^t \big\langle \widetilde{f}_j, v_j'(s)\big\rangle ds \\ &- 2\widetilde{K}_j \int_0^t \big\langle |u|^{p-2}u, v_j'(s)\big\rangle ds - 2\widetilde{\lambda}_j \int_0^t \big\langle |u'|^{q-2}u', v_j'(s)\big\rangle ds \\ &+ 2\int_0^t \widehat{g}_j(s)v_j'(1,s)ds - 2\int_0^t \widetilde{\mu}_j(s)\big\langle u_{jx}(s), v_{jx}'(s)\big\rangle ds \\ &- 2K_j \int_0^t \big\langle |u_j|^{p-2}u_j - |u|^{p-2}u, v_j'(s)\big\rangle ds \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} S_j(t) =& \|v_j'(t)\|^2 + \mu(t)\|v_{jx}(t)\|^2 + K_1(t)|v_j(1,t)|^2 \\ &+ 2\int_0^t \lambda_1(s)|v_j'(1,s)|^2 ds. \end{aligned} \tag{3.16}$$

Using the inequalities (2.12), (3.8), (3.9) and

$$S_j(t) \geq \|v_j'(t)\|^2 + \mu_0\|v_{jx}(t)\|^2 + 2\lambda_0 \int_0^t |v_j'(1,s)|^2 ds, \tag{3.17}$$

then, we can prove the following inequality in a similar manner

$$\begin{aligned}S_j(t) \leq &\frac{\beta}{\lambda_0}S_j(t) + \frac{1}{\beta}\|\widehat{g}_j\|^2_{L^2(0,T)} + \|\widetilde{f}_j\|^2_{L^2(Q_T)} + \frac{1}{\mu_0}TM_T\|\widetilde{\mu}_j\|^2_\infty \\ &+ T\bigg(\frac{M_T}{\mu_0}\bigg)^{p-1}\big|\widetilde{K}_j\big|^2 + T\bigg(\frac{M_T}{\mu_0}\bigg)^{q-1}\big|\widetilde{\lambda}_j\big|^2 \\ &+ \int_0^t \bigg[4 + \|\mu'\|^2_\infty + \frac{1}{\beta\mu_0}T\|k\|^2_{L^2(0,T)} \\ &\qquad + \frac{2K^\star}{\sqrt{\mu_0}}(p-1)R^{p-2} + |K_1'(s)|\bigg]S_j(s)ds,\end{aligned} \tag{3.18}$$

for all $\beta > 0$ and $t \in [0, T]$.
Choosing $\beta > 0$ such that $\frac{\beta}{\lambda_0} \leq \frac{1}{2}$ and denoting

$$\begin{aligned}\widetilde{R}_j =&\frac{2}{\beta}\|\widehat{g}_j\|^2_{L^2(0,T)} + 2\big\|\widetilde{f}_j\big\|^2_{L^2(Q_T)} + \frac{2}{\mu_0}TM_T\|\widetilde{\mu}_j\|^2_\infty \\ &+ 2T\bigg(\frac{M_T}{\mu_0}\bigg)^{p-1}|\widetilde{K}_j|^2 + 2T\bigg(\frac{M_T}{\mu_0}\bigg)^{q-1}|\widetilde{\lambda}_j|^2,\end{aligned} \tag{3.19}$$

$$\phi(s) = 2\bigg[4 + \|\mu'\|^2_\infty + \frac{1}{\beta\mu_0}T\|k\|^2_{L^2(0,T)} + \frac{2K^\star}{\sqrt{\mu_0}}(p-1)R^{p-2} + |K_1'(s)|\bigg]. \tag{3.20}$$

Then from (3.18)-(3.20) we have

$$S_j(t) \leq \widetilde{R}_j + \int_0^t \phi(s)S_j(s)ds. \tag{3.21}$$

By Gronwall's lemma, we obtain from (3.21) that

$$S_j(t) \leq \widetilde{R}_j exp\bigg(\int_0^t \phi(s)ds\bigg) \leq D_T^{(1)}\widetilde{R}_j, \quad \forall t \in [0, T], \tag{3.22}$$

where $D_T^{(1)}$ is a positive constant.
On the other hand, using the imbedding $H^1(0,T) \hookrightarrow C^0\big([0,T]\big)$, it follows from (3.13), (3.14), (3.17), (3.19) and (3.22) that

$$\begin{aligned}\|P_j\|_{L^2(0,T)} \leq &\bigg(\sqrt{\frac{T}{\mu_0}}\|K_1\|_\infty + \frac{1}{\sqrt{2\lambda_0}}\|\lambda_1\|_\infty \\ &\qquad + \sqrt{\frac{T}{\mu_0}}\|k\|_{L^2(0,T)}\bigg)\sqrt{D_T^{(1)}\widetilde{R}_j} + \|\widehat{g}_j\|_{L^2(0,T)},\end{aligned} \tag{3.23}$$

$$
\begin{aligned}
\widetilde{R}_j \le{}& \frac{2}{\beta}\|\widehat{g}_j\|^2_{L^2(0,T)} + 2\|\widetilde{f}_j\|^2_{L^2(Q_T)} + \frac{2}{\mu_0}TM_T\|\widetilde{\mu}_j\|^2_{H^1(0,T)} \\
&+ 2T\left(\frac{M_T}{\mu_0}\right)^{p-1}\left|\widetilde{K}_j\right|^2 + 2T\left(\frac{M_T}{\mu_0}\right)^{q-1}\left|\widetilde{\lambda}_j\right|^2 \\
\le{}& D_T^{(2)}\left(\|\widehat{g}_j\|^2_{L^2(0,T)} + \|\widetilde{f}_j\|^2_{L^2(Q_T)} + \|\widetilde{\mu}_j\|^2_{H^1(0,T)} + \left|\widetilde{K}_j\right|^2 + \left|\widetilde{\lambda}_j\right|^2\right),
\end{aligned}
\tag{3.24}
$$

$$
\begin{aligned}
\|\widehat{g}_j\|_{L^2(0,T)} \le{}& \|\widetilde{g}_j\|_{H^1(0,T)} + \sqrt{\frac{TM_T}{\mu_0}}\|\widetilde{K}_{1j}\|_{H^1(0,T)} \\
&+ \sqrt{\frac{M_T}{2\lambda_0}}\|\widetilde{\lambda}_{1j}\|_{H^1(0,T)} + \sqrt{\frac{TM_T}{\mu_0}}\|\widetilde{k}_j\|_{H^1(0,T)} \\
\le{}& D_T^{(3)}\Big(\|\widetilde{g}_j\|_{H^1(0,T)} + \|\widetilde{K_{1j}}\|_{H^1(0,T)} \\
&+ \|\widetilde{\lambda}_{1j}\|_{H^1(0,T)} + \|\widetilde{k}_j\|_{H^1(0,T)}\Big).
\end{aligned}
\tag{3.25}
$$

Finally, by (3.2), (3.11) and the estimates (3.22)-(3.25), we dedude that (3.3) holds. Hence, Theorem 2 is proved thoroughly. □

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